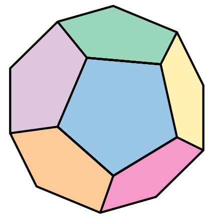
November ARML Solutions



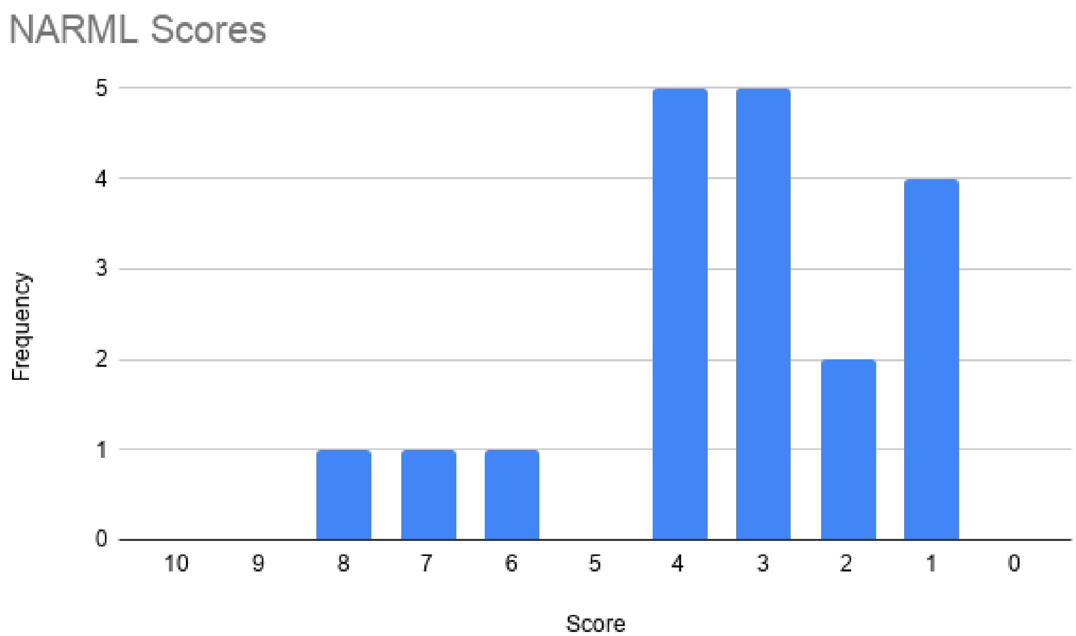
Answer Key

The proposer for each problem is indicated next to the answer.

1. 41 (skyscraper)
2. 8 (skyscraper)
3. $0, 2, 8$ (dchenmathcounts)
4. 40 (skyscraper)
5. $\frac{22}{3}$ (dchenmathcounts)
6. 624 (dchenmathcounts)
7. $\frac{2\sqrt{145}}{5}$ (youyanli)
8. $\frac{292}{2113}$ (dchenmathcounts)
9. 48 (vvluo)
10. $\sqrt{3}+\sqrt{2}$ (Depsilon0)

Statistics

Thanks to all 22 people for submitting the contest. The average score was 2.95. Problem statistics are shown below.


NARML Scores
Frequency
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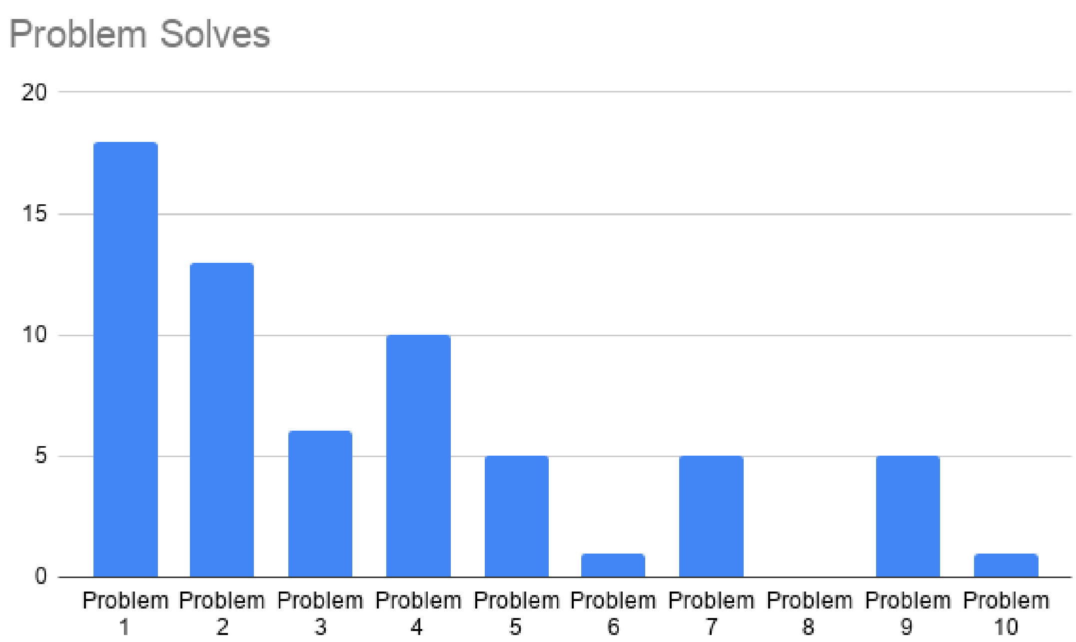
Problem Solves
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Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Problem 6
Problem 7
Problem 8
Problem 9
Problem 10

MAC

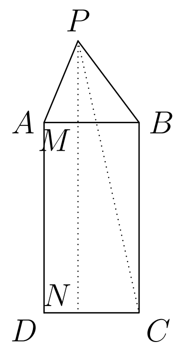
Problem 1

Problem 1

Suppose $ABCD$ is a rectangle with $AB = 14$ and $BC = 28$. If point P outside $ABCD$ satisfies the conditions $PA = 13$ and $PB = 15$, compute the length of PC.

Solution

Note the distance from P to $\overline{AB}$ is 12, and let the perpendicular from P to $\overline{AB}$ meet $\overline{AB}$ at point M and $\overline{CD}$ at point N. Note that $PM = 12$ as $\triangle PAB$ is a $13-14-15$ triangle, so $BM = CN = 9$ and $PN = PM + MN = PM + BC = 12 + 28 = 40$. Thus by the Pythagorean Theorem, $PC = \sqrt{9^2 + 40^2} = \mathbf{41}$.



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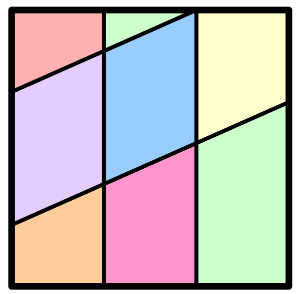
Problem 2

Problem

The numbers $\{1, 2, \ldots, 8\}$ are placed in each of the cells in the magic square below such that the number in each cell is distinct, and the sum of all numbers in each slanted row and each column is the same. Compute the number of ways that the magic square can be filled out.



Solution

Note that the sum of the numbers in each row or column is $\frac{1+2+\cdots+8}{3} = 12$.

Now note that 8 cannot be in a row with 3 cells and a column with 3 cells, since the only way to express 4 as a sum of two distinct numbers is $1 + 3$. So 8 must be in a row or column with two cells, and there are 4 ways to satisfy this condition. Also note that this determines the position of 4.

We further claim that similarly 7 must also be in a row or column with 2 cells. Note that the only ways to express 5 as the sum of two numbers are $1 + 4$ and $2 + 3$, but since 4 is used because of the 8 there is only one sum that works. So there are 2 ways to place 7, and this determines the position of 5 as well.

Now we show that all 8 arrangements of $4, 5, 7, 8$ create a unique arrangement of numbers. Note that 1 must be in the same three-cell group as 8 and the same three-cell group as 5, which uniquely determines 1, and that a similar argument can be made for $2, 3, 6$. Thus there is a unique arrangement of $1, 2, 3, 6$ for each arrangement of $4, 5, 7, 8$. Thus, there are $4 \cdot 2 = \mathbf{8}$ ways to fill out the magic square.

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Problem 3

Problem

Determine all values of a such that the equation

$$ax^2 - (a+4)x + \frac{9}{2} = 0$$

only has one real solution over x.

Solution

If this is a quadratic equation, then we must have the discriminant be 0, or $(a+4)^2 - 18a = a^2 - 10a + 16 = (a-2)(a-8) = 0$. Thus the solutions for this case are $a = 2, a = 8$.

But don't forget that this system only has one solution if it is linear, or $a = 0$. So the entire solution set is $\mathbf{0, 2, 8}$.

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Problem 4

Problem

Compute the smallest positive integer n such that $9(n+3)$ divides $4n!+n+5$.

Solution

Note that $n \equiv 4 \pmod 9$ since $9 \mid 4n!+n+5$. We claim that $n+3 \mid 4n!+n+5$ if and only if $n+3$ is prime. Note that if $n+3$ is not prime then $n+3 \mid n!$ and if $n+3$ is prime, $4n!+n+5 \equiv 4n!+2 \equiv \frac{4}{n-2}+2 \equiv -2+2 \equiv 0 \pmod n$ by Wilson's Theorem.

Now we claim that $9 \mid 4n!+n+5$ if and only if $n \equiv 4 \pmod 9$. Note that if $n \geq 6$ then obviously this is true as $9 \mid n!$, and if $3 \leq n \leq 6$ the only possible value of n is 4 for mod 3 reasons, which can be verified to not work. Now we can check directly that $1 \leq n \leq 2$ has no solutions.

Now we check the positive integers congruent to 4 mod 9 and find that **40** is the smallest number that satisfies both conditions.

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Problem 5

Problem

Let $a_1, a_2, a_3, \ldots$ be a sequence that satisfies $a_1 = a_2 = 1$ and $4a_n = 9a_{n-2} - a_{n-1}$. Compute

$$\sum_{n=1}^{\infty} a_n \cdot \left(\frac{2}{3}\right)^n.$$

Solution 1

Note that

$$a_n \left(\frac{2}{3}\right)^n = (9a_{n-1} - 4a_{n+1}) \left(\frac{2}{3}\right)^n = 6\left(a_{n-1}\left(\frac{2}{3}\right)^{n-1} - a_{n+1}\left(\frac{2}{3}\right)^{n+1}\right).$$

Let $b_n = a_n \left(\frac{2}{3}\right)^n$, and notice this sum decomposes into

$$b_1 + 6\left((b_1 - b_3) + (b_3 - b_5) + \cdots\right) + 6\left((b_2 - b_4) + (b_4 - b_6) + \cdots\right) = b_1 + 6(b_1 + b_2) = \frac{2}{3} + 6\left(\frac{2}{3} + \frac{4}{9}\right) = \mathbf{\frac{22}{3}}.$$

Solution 2

Let $b_n = a_n(\frac{2}{3})^n$. Plugging this into the given recursion, we get $b_n = b_{n-2} - \frac{b_{n-1}}{6}$. Let $f(x) = b_1 + b_2 x + b_3 x^2 \cdots$. Then, we have $(x^2 - \frac{x}{6})f(x) = f(x) - b_1 - b_2 - \frac{x}{6}b_1$. Plug in $b_1 = \frac{2}{3}, b_2 = \frac{4}{9}$ and $x = 1$ to get $f(1) = \mathbf{\frac{22}{3}}$.

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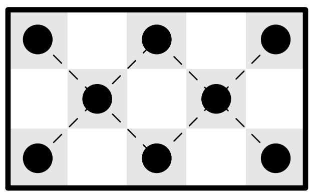
Problem 6

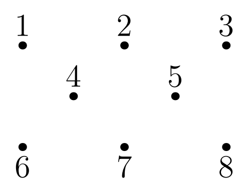
Problem

Consider a 3×5 rectangle colored in a checkerboard pattern, with its corner squares being black. Rocks of different colors are put on each black square. A valid move consists of taking a stack of rocks and placing it over a diagonally adjacent square with at least one rock on it. If all of the rocks end up in the same stack, how many ways can the rocks in the stack be ordered?



Solution

Label the rocks as shown in the diagram below.



Call $1, 3, 6, 8$ corner rocks, $2, 7$ side rocks, and $4, 5$ center rocks. We do casework on which position has the final stack.

Case 1: Corner Rock
We have 4 corners to pick from, so without loss of generality have the stack end up at 1 and multiply by 4 afterwards. Note that this is equivalent to stacking all of the rocks except for 1 on 4. We now take subcases based on how many rocks move directly to 4.

Subcase 1.1: Two moves to 4
Note that one of these must obviously be 6. Thus it can insert itself in 2 positions, so we can ignore 6 and multiply by 2 at the end.

Now there are two ways to choose which one of 2 and 7 goes to 4. Without loss of generality, say that 2 goes to 4. Then there are $3! = 6$ ways to determine the order of which $3, 7, 8$ go on 5. Thus there are $2 \cdot 2 \cdot 6 = 24$ ways to stack the rocks.

Subcase 1.2: Three moves to 4

Note that one of these must obviously be 6. Thus it can insert itself in 3 positions, so we can ignore 6 and multiply by 3 at the end.

Now there are two ways to decide which order $3, 8$ go onto 5. Now 5 can go onto either 2 or 7, and there are 2 ways to pick the order of which 2 and 7 go onto 4. Thus there are $3 \cdot 2 \cdot 2 \cdot 2 = 24$ ways to stack the rocks.

In total, there are $4 \cdot (24 + 24) = 192$ ways to stack the rocks if the stack ends on a corner.

Case 2: Side Rock
We have 2 side rocks to pick from, so without loss of generality let the stack end up at 2 and multiply by 2 afterwards. We take subcases based on how many rocks move directly to 2.

Subcase 2.1: One move to 2
We can pick whether it is 4 or 5 that ends up moving to 2 at the end. Without loss of generality, let the penultimate rock be 4, and multiply by 2 at the end. Note that there are 2 ways to stack $3, 8$ on 5, that 5 is forced to 7, and that there are $3! = 6$ ways to stack $1, 6, 7$ on 4, so the total number of ways is $2 \cdot (2 \cdot 6) = 24$.

Subcase 2.2: Two moves to 2
Note that there are 2 ways to pick which one of $4, 5$ that 7 goes to. Without loss of generality, let 7 go to 4, and multiply by 2 at the end. Note that there are $3!$ ways to order $1, 6, 7$ on 4, two ways to order $3, 8$ on 5, and two ways to order $4, 5$ on 2. Thus there are $2 \cdot (3! \cdot 2 \cdot 2) = 48$ ways to stack the rocks.

In total there are $2 \cdot (24 + 48) = 144$ ways to stack the rocks if the stack ends on a side.

Case 3: Center Rock
We have 2 center rocks to pick from, so without loss of generality let the stack end up at 4 and multiply by 2 afterwards. We take subcases based on how many rocks move directly to 4.

Subcase 3.1: Three moves to 4
One of $2, 7$ must move onto 4, so without loss of generality, let 2 move to 4 and multiply by 2 at the end. Note that there are $3! = 6$ ways to stack $3, 7, 8$ on 5, that 5 is forced to 2, and that there are $3! = 6$ ways to order $2, 6, 7$. Thus there are $2 \cdot (6 \cdot 6) = 72$ ways to stack the rocks.

Subcase 3.2: Four moves to 4
There are 2 ways to stack $3, 8$ on 5, 2 places for 5 to go (it can go to 2 and 7), and $4! = 24$ ways to stack $1, 2, 6, 7$ on 4. Thus there are $2 \cdot 2 \cdot 24 = 72$ ways to stack the rocks.

In total there are $2 \cdot (72 + 72) = 288$ ways to stack the rocks.

Thus our total is $192 + 144 + 288 = \mathbf{624}$.

Problem 7

Problem

Consider $\triangle ABC$ with a right angle at B and $AB = 2$. Let point $D \neq C$ lie on segment AC with $AD = 4$, and let E be a point such that segment DE is a diameter of the circumcircle of $\triangle BDC$. If $AE = 10$, compute AC.

Solution

Note that $\triangle ABC \sim \triangle DBE$, so there is a spiral similarity sending $AC \to DE$ about B, implying that $\triangle ABD \sim \triangle CBE$. Note that if $BC = x$, then $CE = 2x$. Now note $AC = \sqrt{x^2+4}$ by the Pythagorean Theorem. Then $(2x)^2 + (\sqrt{x^2+4})^2 = 100$, implying $5x^2 = 96$, or

$$\sqrt{x^2+4} = \sqrt{\frac{116}{5}} = \mathbf{\frac{2\sqrt{145}}{5}}.$$

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Problem 8

Problem

The mad scientist Kyouma is traveling on a number line from 1 to 2020, subject to the following rules:

- He starts at 1.
- Each move, he randomly and uniformly picks a number greater than his current number to go to.
- If he reaches 2020, he is instantly teleported back to 1.
- There is a time machine on 199.
- A foreign government is waiting to ambush him on 1729.

What is the probability that he gets to the time machine before being ambushed?

Solution

Say Kyouma is in state A if he is between 1 and 198 inclusive or between 1730 and 2020 inclusive, and say he is in state B if he is between 199 and 1729 inclusive. We note that the only way Kyouma can win or lose is by going from state A to state B. Let $P(A)$ be the probability that he wins while in state A, and let $P(B)$ be the probability that he wins while in state B **while the game is still going**. Note that

$$P(A) = \frac{1}{1729-199+1} + \frac{1729-199-1}{1729-199+1}P(B), \text{ and}$$
$$P(B) = (1 - \frac{1}{2020-1729+1})P(A),$$

since we only care about the moves from A to B, and vice versa.

Now we just need to solve this equation. Note that this is equivalent to

$$P(A) = \frac{1}{1531} + \frac{1529}{1531}P(B), \text{ and}$$
$$P(B) = \frac{291}{292}P(A),$$

which implies $1531P(A) = 1 + 1529 \cdot \frac{291}{292}P(A)$, or

$$(2 + 1529 \cdot \frac{1}{292})P(A) = \frac{2113}{292}P(A) = 1,$$

so the answer is $\mathbf{\frac{292}{2113}}$ as we are looking for $P(A)$.

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Problem 9

Problem

Let $\mathcal{H}(x)$ be a function on the positive integers such that

- $\mathcal{H}(1) = 1$, and
- for integers $n > 1$, $2\mathcal{H}(n) = \sum_{i|n} \mathcal{H}(i)$.

Compute the smallest positive integer $n > 1$ that satisfies $\mathcal{H}(n) = n$.

Solution

Note that $\mathcal{H}(x)$ sums the $\mathcal{H}$ values for proper divisors of x. The key is to realize that $\mathcal{H}(n)$ only depends on the structure of its prime factorization, not the primes themselves, since $\mathcal{H}(p) = 1$ for any prime p.

Now, we evaluate $\mathcal{H}$ for numbers in the form of p, p^2, p^3, p^4, p^5, pq, p^2q, p^3q, p^4q, p^2q^2, and pqr for primes p, q, and r.

One can confirm quickly using the values of proper divisors that $\mathcal{H}(p^k) = 2^{k-1}$ for any k defined, $\mathcal{H}(pq) = 3$, $\mathcal{H}(p^2q) = 8$, $\mathcal{H}(p^3q) = 20$, and $\mathcal{H}(p^4q) = 48$. We also have $\mathcal{H}(p^2q^2) = 26$ and $\mathcal{H}(pqr) = 13$. For instance, we have $\mathcal{H}(p^2q) = \mathcal{H}(pq) + \mathcal{H}(q) + \mathcal{H}(p^2) + \mathcal{H}(p) + \mathcal{H}(1) = 3 + 1 + 2 + 1 + 1$.

Thus, we realize that $p^4q =$ **48** $= \mathcal{H}(p^4q)$ is the first such number to satisfy $\mathcal{H}(n) = n$, as $p = 2$ and $q = 3$ works. It is the smallest such number because all the positive integers(not named 1) less than 48 fall into one of the forms we evaluated, and none of them satisfy $\mathcal{H}(n) = n$.

Remark: The motivation behind the solution is that after realizing the symmetry across primes, testing common prime factorization structures for relatively small numbers comes next.

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Problem 10

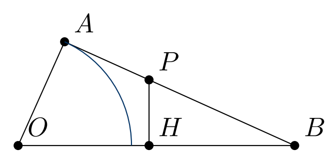
Problem

Three mutually tangent spheres with radii of 1 are tangent to a table, and a cone is tangent to all three spheres with its tip oriented towards the table. If the cone has height $\sqrt{2}$ and its tip is $\frac{3+\sqrt{3}}{3}$ units above the table, compute the radius of the cone.

Solution

Let O be the center of one of the spheres, H be the center of the triangle formed by the centers of the spheres, and P be the tip of the cone. Note that $PH \perp OH$ by symmetry, and take the cross section of one of the spheres and the cone with plane OPH.

Now label the diagram as follows, where AP is tangent the sphere and AP intersects OH at B.



Now note $OH = \frac{2\sqrt{3}}{3}$ and $PH = \frac{\sqrt{3}}{3}$, by the conditions given to us in the problem. By the Pythagorean Theorem, $OP = \frac{\sqrt{15}}{3}$ and thus $AP = \frac{\sqrt{6}}{3}$. Now let $BH = x$, $BP = y$, and note that $\triangle BHP \sim \triangle BAO$, implying that

$$\frac{x}{\frac{\sqrt{3}}{3}} = y + \frac{\sqrt{6}}{3}$$

$$\frac{y}{\frac{\sqrt{3}}{3}} = x + \frac{2\sqrt{3}}{3}.$$

Now note that this implies

$$3x = \sqrt{3}y + \sqrt{2} = x + \frac{2\sqrt{3}}{3} + \sqrt{2}$$

$$x = \frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{2}.$$

Note that by similar triangles, $\frac{x}{PH} = \frac{r}{\sqrt{2}}$, where r is the radius of the sphere. Therefore

$$r = \sqrt{2} \cdot \frac{\frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{3}} = \boldsymbol{\sqrt{2} + \sqrt{3}}.$$

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