2021 JMC 12 Solutions

Mathematical Advancement Committee

February 1, 2021

1. **Answer: (D)**

   Note that $2121^2 - 2120^2 = 2121 + 2120$ and $2020^2 + 2021^2 = 2020 + 2021$ by difference of squares. So, the desired answer is equal to $(2121 + 2120) - (2021 + 2020) = 200$.

2. **Answer: (B)**

   The month mm and date dd cannot be of the form 00. Each digit m and d can be either 0 or 1, but we must subtract the undesired case of 00. So, mm only has $2^2 - 1 = 3$ possibilities, and dd also has $2^2 - 1 = 3$ possibilities. In total, there are $3^2 = 9$ binary days in a year.

3. **Answer: (B)**

   Let x be the amount of aluminum removed, in grams. There are $96 - x$ grams of aluminum left and $100 - x$ grams of the mixture left. So, $\frac{96-x}{100-x} = \frac{9}{10} \implies x = 60$. The mixture has $100 - 60 = 40$ grams.

4. **Answer: (C)**

   Note that $(|x+1| - |x-1|)^2$ equals $(2x)^2$ when $x \in (-1, 1)$ and is equal to $2^2 = 4$ elsewhere. Graphing the portion of $(|x+1| - |x-1|)^2$ with $x+3$, we have two intersections, both of which lie on the parabola $y = 4x^2$. Thus, we set $y = 4x^2 = x + 3$ and get $4x^2 - x - 3 = 0$, which has roots of sum $\frac{1}{4}$ by Vieta's.

5. **Answer: (A)**

   Suppose an odd pretentious three-digit number is of the form $\underline{a}\ \underline{b}\ \underline{c}$, where c equals $1, 3, 5, 7$, or 9. Both a and b can be even, be even and odd, odd and even, but cannot both be odd. Using complementary counting, there are $9 \cdot 10 = 90$ total choices for (a, b) and $5 \cdot 5 = 25$ undesired cases (when both digits are odd), leaving $90 - 25 = 65$ desired pairs (a, b). Thus, the answer is $65 \cdot 5 = 325$ such numbers.

6. **Answer: (A)**

   By Pythagoras, $DX = \sqrt{5^2 - 4^2} = 3 \implies CX = CD - DX = 4 - 3 = 1$. Furthermore, $\angle BAY = \angle CYD$, implying that $\triangle YBA \sim \triangle YXC$. Let $YB = x$. Because of the similarity, $\frac{1}{4-x} = \frac{x}{4} \implies x = 2$. So $YC = 4 - x = 2$, and by Pythagoras, $DY = \sqrt{5}$, and $AY = 2\sqrt{5}$. The answer is $2\sqrt{5} \cdot \sqrt{5} = 10$.

7. **Answer: (D)**

   Note that k must be of the form $3 \cdot 2^a \cdot 5^b$ where $a = 0, 1, 2$ and $b = 0, 1$. To find this, observe that 3 must divide k. Suppose that $k = 3x$. This implies that $x = \gcd(k, 20)$, so x must be a divisor of 20, confirming what we noted. The sum of all k equals $3(1 + 2 + 4)(1 + 5) = 126$.

8. **Answer: (B)**

   Let A be the angle chosen from $1°, 2°, \ldots, 90°$ and B be the angle chosen from $1°, 2°, \ldots, 89°$. For the triangle to be obtuse, we must have $180° - A - B > 90° \implies A + B < 90°$.

   Suppose $A = n°$. Then, B can equal $1°, 2°, \ldots (90 - n - 1)°$. We can see that there are $1 + 2 + \cdots + 88 = \frac{88 \cdot 89}{2} = 44 \cdot 89$ desired cases and $89 \cdot 90$ total cases, so the answer is $\frac{44 \cdot 89}{89 \cdot 90} = \frac{22}{45}$.

9. **Answer: (D)**

We carefully track Emma's positions immediately after a time t with the function $f(t) = (X, p)$ where X is the line Emma is in (A or B) and p is Emma's position in line. Now listing suffices; $f(0) = (A, 1), f(1) = (B, 6), f(2) = f(3) = (B, 4), f(4) = f(5) = (B, 2), f(6) = (A, 2), f(7) = (B, 5), f(8) = f(9) = (B, 3), f(10) = f(11) = (B, 1), f(12) = (A, 3), f(13) = (A, 1)$. Our answer is 13.

10. **Answer: (C)**

Recall that $\sin^2 \alpha + \cos^2 \alpha = 1$. By Vieta's, we have $\sin \alpha + \cos \alpha = n$ and $\sin \alpha \cos \alpha = \frac{1}{4}$. It follows that $\sin^2 \alpha + \cos^2 \alpha = n^2 - 2 \sin \alpha \cos \alpha = n^2 - \frac{1}{2} = 1 \implies n = \frac{\sqrt{6}}{2}$, as desired.

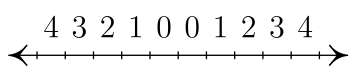
11. **Answer: (E)**

Note that a_i has exactly $i + 1$ ones and the rest are zeroes. By rules of divisibility by 9, the sum of the digits must be a multiple of 9, so we must have $9|(i + 1)$, which have a_i divisible by 9.

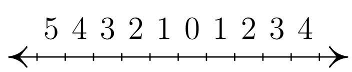
For divisibility of 11, we claim that $11|a_i$ if and only if i is odd. Notice that $11, 1001, 100001, \ldots$ are all divisible by 11 by the sum of odd powers factorization of 1 and 10. Because a_i for odd i are just linear combinations of these, they are multiples of 11. Because 1 more than a multiple of a multiple of 11 is not a multiple of 11, even i fail on the basis of odd i's success. Combining, we must have $(n + 1) \mid 18$, and the answer is the number of $n = 1, 2, \ldots, 2021$ that are congruent to 17 (mod 18). Thus, the answer is $\lfloor \frac{2021-1}{18} \rfloor = 112$.

12. **Answer: (B)**

We use an area-based approach along with number lines, where are total area is $10 \cdot 10 = 100$. We can stop worrying about what happens when at least one of a and b is an integer, because those cases are negligible. Below is the number line representing the value of $\lfloor |a| \rfloor$ in the intervals. If seeing what happens on $a \in (n, n + 1)$ is too complicated, plugging in $a = n + \frac{1}{2}$ will suffice.



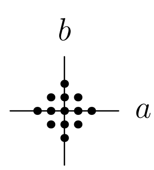
Below is the number line representing the value of $|\lfloor b \rfloor|$ in the intervals.



We seek the number of $10 \cdot 10$ pairs of a and b intervals where $\lfloor |a| \rfloor = |\lfloor b \rfloor|$. Doing simple casework on the equal value, we have $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 0 \cdot 1 = 18$ cases, so our answer is $\frac{18}{10 \cdot 10} = \frac{9}{50}$.

13. **Answer: (E)**

We first look at the upper bound of $|a| + |b| + |c| \leq 5$ only. Consider the three-dimensional space for (a, b, c). If $c = -5$, notice that the ab cross-section contains only one solution, $(0, 0)$. If $c = -4$, notice that we need $|a| + |b| \leq 1$, so graphing this out on the ab plane yields 5 solutions. Specifically, $5 = 1 + 3 + 1$ if we go b-coordinate by b-coordinate. Similarly, $c = -3 \implies 13 = 1 + 3 + 5 + 3 + 1$ solutions, as shown below.



Positive and negative c have the same number of solutions. Summing all of this together, we have $2((0+1)+(1+4)+(4+9)+(9+16)+(16+25))+(25+36) = 231$ solutions if we only consider the upper bound. We can apply the same logic to see that there are 7 solutions to $|a|+|b|+|c| < 2$. Thus, our answer is $231 - 7 = 224$.

14. **Answer: (E)**

Note that for the quadrilateral to have positive area,

$$\log 3(\log_{a_1} a_2) + \log 3(\log_{a_2} a_3) + \log 3(\log_{a_3} a_4) = \log 27(\log_{a_1} a_4) > \log 3(\log_{a_4} a_1)$$

$$\implies 27(\log_{a_1} a_4) > 3(\log_{a_4} a_1)$$

Since $\log_{a_1} a_4 = (\log_{a_4} a_1)^{-1}$, we have $(\log_{a_4} a_1)^2 < 9$, or $\log_{a_4} a_1 < 3$. Also note that in order for $\log 3(\log_{a_4} a_1)$ to be positive, we must have $\log_{a_4} a_1 > \frac{1}{3}$. Similarly, we also have $\frac{1}{3} < \log_{a_1} a_2, \log_{a_2} a_3, \log_{a_3} a_4 < 3$. In addition, we must have $\log_{a_1} a_2 \cdot \log_{a_2} a_3 \log_{a_3} a_4 \cdot \log_{a_4} a_1 = 1$. Here, we see that $p = 9$, which $\log_{a_1} a_3$ can be arbitrarily close to by letting $\log_{a_1} a_2 = \log_{a_2} a_3 = 3 - \epsilon$ and $\log_{a_3} a_4 = \log_{a_3} a_4 = \frac{1}{3} - \epsilon$ for arbitrarily small ϵ. Thus $q = \frac{1}{9}$ and our answer is 81.

15. **Answer: (C)**

Note $\angle YPX = \angle BPC = 120^\circ$, implying that quadrilateral $AXPY$ is cyclic. Thus, $\angle PBA = \angle PAB = \angle PAY = \angle PXY = 15^\circ$. We also have $\angle PCA = \angle PAC = 60^\circ - 15^\circ = 45^\circ$, so $\triangle PAC$ is an isosceles right triangle. Because $AXPY$ is cyclic, $\triangle PAC \sim \triangle XYC \implies XY = \frac{CY}{\sqrt{2}} = 3\sqrt{2}$.

16. **Answer: (B)**

Observe that

$$(d_1 - d_2)|d_1 d_2 \implies (k-1)d_2|k \cdot (d_2)^2 \implies (k-1)|kd_2.$$

Because $\gcd(k, k-1) = 1$, it follows that $(k-1)|d_2$. Since $d_2|1296$, we must also have $(k-1)|1296$. Note that we cannot have $6|(k-1)$, because this will result in k being neither a multiple of 2 nor 3.

Thus, we need only to check $(k-1)|16$ and $(k-1)|81$. Note that k must be of the form $2^a \cdot 3^b$ for non-negative integers a and b, so our desired answer is $2+3+4+9 = 18$.

17. **Answer: (C)**

Putting terms of different degrees on different sides, we can factor to obtain;

$$\begin{cases} 96(y+z) = (x+y)(x+z) \\ 24(x+z) = (y+z)(x+y) \\ 54(x+y) = (x+z)(y+z) \end{cases}$$

Multiplying these and simplifying yields $(x+y)(y+z)(x+z) = 96 \cdot 24 \cdot 54$, and substituting gives

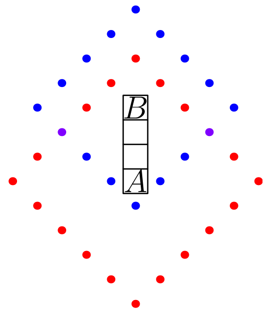
$$96(y+z) = \frac{96 \cdot 24 \cdot 54}{(y+z)} \implies (y+z)^2 = 24 \cdot 54 = 1296 \implies y+z = 36.$$

Similarly, $x+z = 72$ and $x+y = 48$. So $(x,y,z) = (42,6,30)$ and $xyz = 42 \cdot 6 \cdot 30 = 7560$, as desired.

18. **Answer: (D)**

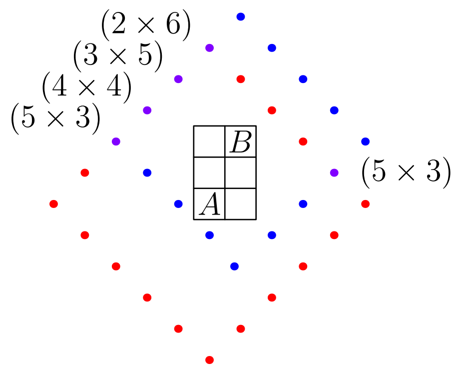
We do cases on whether A to B is three steps in one direction or two steps in one direction and one step in a perpendicular direction.

Case 1. Three steps straight from A to B.



The red dots mark the places of squares that are 5 away from A and the blue dots mark the squares that are 4 away from B. The two purple dots are valid placements for C. However, we can also have A on top and B on bottom, or A and B aligned horizontally, so we have 4 symmetries. There are also $2 \cdot 2$ ways to embed the 4×4 hull of ABC into our 5×5 grid, so our total for this case is $4 \cdot 4 \cdot 2 = 32$.

Case 2. Two steps one direction, one step in a perpendicular direction from A to B.



We can embed the 2×6 hull in 0 ways (we only have a 5×5 grid), a 3×5 or 5×3 hull in 3 ways, and a 4×4 in $2 \cdot 2 = 4$ ways. This gives $1 \cdot 0 + 3 \cdot 3 + 1 \cdot 4 = 13$ ways. However, we must multiply this by 8 to accommodate for the 8 positions (symmetric) of B relative to A, giving $13 \cdot 2^3 = 104$ cases here. Our final answer is $104 + 32 = 136$.

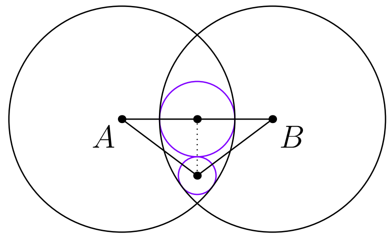
19. **Answer: (C)**

Let A and B be the centers of ω_a and ω_b respectively. Let $R = 1$ be the radius of ω_a and ω_b and $D = \frac{4}{3}$ be the distance between A and B. Note that the centers of ω_1 and ω_2, say O_1 and O_2 respectively, lie on a line that is both perpendicular to AB and equidistant from A and B.

Because $\triangle AO_1O_2 \cong \triangle BO_1O_2$, we have that $\frac{D}{2}$ is the length of the A-altitude of AO_1O_2. We have $AO_1 = R - r_1$, $AO_2 = R - r_2$, and $O_1O_2 = r_1 + r_2$, so $\triangle AO_1O_2$'s perimeter is $2R$. Thus, by Heron's Formula $[\triangle AO_1O_2] = \sqrt{Rr_1r_2(R - r_1 - r_2)} = \frac{1}{2} \cdot \frac{D}{2} \cdot (r_1 + r_2)$. Substituting known values, we have

$$\sqrt{1 \cdot r_1r_2(1 - \tfrac{1}{2})} = \frac{\frac{4}{3}}{4} \cdot \frac{1}{2},$$

whence $r_1 \cdot r_2 = \frac{1}{18}$.



Remark: In this specific case, $\triangle AO_1O_2$ is actually a right triangle with lengths in the ratio $3 : 4 : 5$, which is why the diagram has one of the centers lying on AB.

20. **Answer: (A)**

Note that $\arg(\frac{(z+\overline{z})^2}{\overline{z}}) = \arg(z)$. Suppose this value equals θ. Then $|\frac{(z+\overline{z})^2}{\overline{z}}| = (2\cos\theta)^2 \cdot \theta \cdot |z| = 4\cos^2\theta|z|$. Observe that the foot of the altitude from $z + \overline{z}$ to the line through 0 containing z is $(1 + \cos 2\theta) \cdot z$. However, $1 + \cos 2\theta = 2\cos^2\theta$, so $\frac{(z+\overline{z})^2}{\overline{z}}$ is actually twice this number. So $f(z) = \frac{(z+\overline{z})^2}{\overline{z}} - (z + \overline{z})$ is the reflection of $z + \overline{z}$ across the line through 0 containing z.

It follows that $f(z)$ doubles the acute angle between z and the real axis without flipping over the real axis and $f(z)$ also multiplies the magnitude by $|\frac{z+\overline{z}}{z}| = 2|\cos\theta|$. Thus, $f(z)$ has two solutions, namely $\arg(\frac{2\pi}{3})$ and $\arg(\frac{\pi}{6})$. The magnitudes of the two solutions need to be multiplied by 1 and $\sqrt{3}$ respectively to reach $|1 + i\sqrt{3}| = 2$, so the two solutions are $(\frac{2}{1} \cdot \text{cis}(\frac{2\pi}{3}), \frac{2}{\sqrt{3}} \cdot \text{cis}(\frac{\pi}{6})) \implies \frac{4i\sqrt{3}}{3}$.

21. **Answer: (A)**

The crux of this problem is noting that the anteater could only make forward progress. Let e_i be the life expectancy of the ant where the closest path from the anteater to the ant is a distance of i edges, and $e_0 = 0$. Note that $e_1 = \frac{1}{3} \cdot -\frac{1}{2} + \frac{2}{3}e_1 + 1 \implies e_1 = \frac{5}{2}$. Here, $-\frac{1}{2}$ comes from the death of the ant occurring before the step has gone through completely. Now, for e_2, the anteater chooses one of two congruent paths. We have $e_2 = \frac{1}{3} \cdot 0 + \frac{2}{3}e_2 + 1 \implies e_2 = 3$. Similarly, we obtain $e_3 = \frac{2}{3}e_1 + \frac{1}{3}e_3 + 1 \implies e_3 = 4$. The life expectancy of the ant is $\frac{1}{8}e_0 + \frac{3}{8}e_1 + \frac{3}{8}e_2 + \frac{1}{8}e_3 = \frac{41}{16}$.

22. **Answer: (C)**

Suppose that $a = \sqrt[6]{x}, b = \sqrt[6]{y}$, and $c = \sqrt[6]{z}$. From given conditions, we have

$$\begin{cases} \frac{a^3+b^3+c^3}{3} - abc = \frac{74}{13} \implies a^3 + b^3 + c^3 - 3abc = \frac{222}{13} \\ a^2 + b^2 + c^2 - ab - bc - ca = \frac{37}{9} \end{cases}$$

and we seek $\frac{a+b+c}{3} - \sqrt[3]{abc} = \frac{a+b+c}{3} - 1$.

We can use the special factorization $a^3 + b^3 + c^3 - 3abc = (a^2 + b^2 + c^2 - ab - bc - ca)(a + b + c)$ and substitute the above values to obtain $a + b + c = \frac{54}{13} \implies \mathcal{D}(\sqrt[6]{x}, \sqrt[6]{y}, \sqrt[6]{z}) = \frac{5}{13}$ as desired.

23. **Answer: (B)**

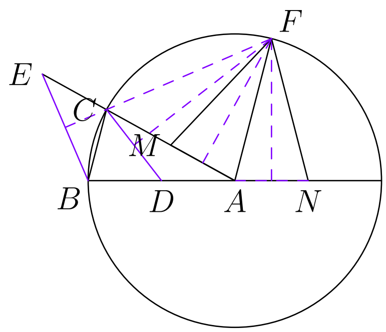
We claim that if the midpoints of segments of equal length AB and CD (not parallel) are M and N respectively, then the perpendicular bisectors of AC, BD, and MN concur at a single point. To prove this, let S be the center of the spiral similarity sending A to C and B to D. The spiral similarity also sends M to N. Since $AB = CD$, it is clear that $SA = SC, SB = SD$, and $SM = SN$.

Let M be the midpoint of AC and N be the reflection of D across A. By applying our claim twice, the perpendicular bisectors of BE, DC, AM, and NA concur at F on ω, implying $FM = FN = FA = 4$.

Meanwhile, $AM = AN = 2$, and it follows that $\triangle FMA$ and $\triangle FAN$ are both isosceles triangles with lengths $4, 4$ and 2. So it follows that

$$\angle BAC = 180^\circ - \angle MAF - \angle FAN = 180^\circ - \angle MAF - \angle AMF = \angle AFM \implies \triangle BAC \cong \triangle AFM$$

Therefore, the area and perimeter are $\sqrt{15}$ and 10 respectively, yielding $\frac{2\sqrt{15}}{3}$.



24. **Answer: (D)**

The problem asks for when $2^{2b} + 11^{2b}$ and $2^a - 3^b$ have the same number of powers of 5.

First, when b is even, the numerator $4^b + 121^b$ is not divisible by 5, so the denominator must also not be divisible by 5. So $2^a \not\equiv (2^3)^b \pmod 5$ if and only if $a \not\equiv 3b \pmod 4$. This gives $75 \cdot 5 = 375$ solutions.

When b is odd, $4^b + 121^b$ has at least 3 powers of 5 due to the sum of b-th powers factorization for odd b. In order for $2^a \equiv 3^b \pmod{125}$, we must have $2^a \equiv (2^7)^b \pmod{125}$. Note that $\text{ord}_2(125) = 100$ (taking 2^{10} and/or 2^{20} modulo 125 will work). Therefore, $7b \equiv a \pmod{100}$. It is clear that $7b = a$ must be satisfied. To verify that all (a, b) with $a = 7b$ work, we check with Lift-the-Exponent Lemma;

$$v_5(121^b + 4^b) = v_5(125) + v_5(b) = v_5(128^b - 3^b)$$

where $v_5(x)$ is the maximum possible k such that 5^k divides x. So when b is odd, there are 5 solutions. Thus, we have in total, 380 solutions.

25. **Answer: (E)**

To construct S and S', let S contain all positive integers with an even number of primes, not necessarily distinct, in its prime factorization. It follows that S' contains all positive integers with an odd number of primes in its prime factorization. To verify that this construction works, let each j be assigned by a function T. So $T(j) = 1$ if j is in S and $T(j) = -1$ if j is in S'. Now, we build up starting with $j = 1$ by assigning the $T(j)$ values. For each j notice that the sum of the $T(i)$ for i being the divisors of j must be 0 for all non-squares and 1 for all perfect squares. Because we know the $T(i)$ values for all proper divisors, we see that $T(j)$, the last value of the sum is uniquely determined for us. Building these $T(j)$ values for small values yields the answer $1 + 4 + 6 + 9 + 10 + 14 = 44$.

Alternatively, to prove to the construction, note that each $j \in \mathbb{Z}_+$ is in exactly one of S or S'. Consider each x^k term starting from $k = 1$ and going up. The first, 1 (which is the only j producing the x^1 term), has to belong in S. Using strong inductive reasoning, we choose whether k is in S or S' when it comes time to judge the count on x^k. Notice that to generate x^k (positive or negative), we must use a j such that $j|k$. Let $k = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$. By our inductive hypothesis, we know whether which divisors of k are in S or S' (except k itself). One may verify that the only way to get our desired coefficient 1 for x^k for perfect squares k and 0 for x^k for non-squares k will imply the inductive hypothesis is true.