2021 JMC 10 Solutions

Mathematical Advancement Committee

February 1, 2021

1. **Answer: (D)**
   We have $\frac{20}{2\cdot 1} - \frac{2+0}{2/1} = 10 - 1 = 9$.

2. **Answer: (B)**
   Note that $e < \pi$ and $2e > \pi$. So, it follows that $|\pi - |e - |e - \pi||| = |\pi - |e - (\pi - e)|| = |\pi - |2e - \pi|| = |2\pi - 2e| = 2\pi - 2e$.

3. **Answer: (D)**
   If one P_i calls P_{i+1} a liar, then only one of $\{P_i, P_{i+1}\}$ can be a liar. If both are liars, we have a contradiction, because a liar would call another liar a truth teller. Likewise, $\{P_i, P_{i+1}\}$ cannot both be truth tellers. We must have an alternating sequence of truth tellers and liars, so there are $\frac{8}{2} = 4$ liars.

4. **Answer: (B)**
   The month mm and date dd cannot be of the form 00. Each digit m and d can be either 0 or 1, but we must subtract the undesired case of 00. So, mm only has $2^2 - 1 = 3$ possibilities, and dd also has $2^2 - 1 = 3$ possibilities. In total, there are $3^2 = 9$ binary days in a year.

5. **Answer: (B)**
   Let x be the amount of aluminum removed, in grams. There are $96 - x$ grams of aluminum left and $100 - x$ grams of the mixture left. So, $\frac{96-x}{100-x} = \frac{9}{10} \implies x = 60$. The mixture has $100 - 60 = 40$ grams.

6. **Answer: (B)**
   Suppose there are N people in the family. After fifteen years, each person's age is increased by 15, and the entire sum of the entire family's age is increase by $15N$. We have $15N + 78 = 153 \implies N = 5$.

7. **Answer: (A)**
   Let the square have side length s. The diagonals have length $s\sqrt{2}$. So, the product of the length of both diagonals is $2s^2$ and the square's area is s^2, which is half of the product of the diagonals. We have $5x + 7 = 2(3x + 1) \implies x = 5$. The area is $3 \cdot 5 + 1 = 16$, implying that the side length is $\sqrt{16} = 4$ and the perimeter is $4 \cdot 4 = 16$.

8. **Answer: (A)**
   Suppose an odd pretentious three-digit number is of the form $\underline{a}\ \underline{b}\ \underline{c}$, where c equals $1, 3, 5, 7$, or 9. Both a and b can be even, be even and odd, odd and even, but cannot both be odd. Using complementary counting, there are $9 \cdot 10 = 90$ total choices for (a, b) and $5 \cdot 5 = 25$ undesired cases (when both digits are odd), leaving $90 - 25 = 65$ desired pairs (a, b). Thus, the answer is $65 \cdot 5 = 325$ such numbers.

9. **Answer: (E)**
   Let e denote our number system and m denote the Malacharian number system. Note that $15_m \cdot 73_m = 51_e \cdot 37_e = 1887_e = 7881_m$, so 7881 is the desired answer.

10. **Answer: (A)**

    By Pythagoras, $DX = \sqrt{5^2 - 4^2} = 3 \implies CX = CD - DX = 4 - 3 = 1$. Furthermore, $\angle BAY = \angle CYD$, implying that $\triangle YBA \sim \triangle YXC$. Let $YB = x$. Because of the similarity, $\frac{1}{4-x} = \frac{x}{4} \implies x = 2$. So $YC = 4 - x = 2$, and by Pythagoras, $DY = \sqrt{5}$, and $AY = 2\sqrt{5}$. The answer is $2\sqrt{5} \cdot \sqrt{5} = 10$.

11. **Answer: (D)**

    Note that k must be of the form $3 \cdot 2^a \cdot 5^b$ where $a = 0, 1, 2$ and $b = 0, 1$. To find this, observe that 3 must divide k. Suppose that $k = 3x$. This implies that $x = \gcd(k, 20)$, so x must be a divisor of 20, confirming what we noted. The sum of all k equals $3(1 + 2 + 4)(1 + 5) = 126$.

12. **Answer: (D)**

    The line $y = 2x$ intersects the curve $y = x^2$ at two points, namely $(0, 0)$ and $(2, 4)$. Note that the line $x + y = n$ must intersect $y = 2x$ at a point strictly between the points $(0, 0)$ and $(2, 4)$ to divide $y \geq x^2$ into four regions. If $x + y = n$ intersects $y = 2x$ at $(0, 0)$, we have $n = 0$. Similarly, if $x + y = n$ intersects $y = 2x$ at $(2, 4)$, we have $n = 6$. So, $0 < n < 6$. The sum of all possible n is equal to 15.

13. **Answer: (B)**

    Let A be the angle chosen from $1°, 2°, \ldots, 90°$ and B be the angle chosen from $1°, 2°, \ldots, 89°$. For the triangle to be obtuse, we must have $180° - A - B > 90° \implies A + B < 90°$.

    Suppose $A = n°$. Then, B can equal $1°, 2°, \ldots (90 - n - 1)°$. We can see that there are $1 + 2 + \cdots + 88 = \frac{88 \cdot 89}{2} = 44 \cdot 89$ desired cases and $89 \cdot 90$ total cases, so the answer is $\frac{44 \cdot 89}{89 \cdot 90} = \frac{22}{45}$.

14. **Answer: (B)**

    Note that $24_b = 2b + 4, 57_b = 5b + 7$, and $72_b = 7b + 2$. Because they are terms in an arithmetic sequence, the difference between 57_b and 24_b must be twice the difference between 72_b and 57_b. It follows that $(5b + 7) - (2b + 4) = 2((7b + 2) - (5b + 7)) \implies b = 13$. So the terms expressed in base 10 are

    $$2 \cdot 13 + 4, n, 5 \cdot 13 + 7, 7 \cdot 13 + 2, \ldots$$

    and $n = \frac{30+72}{2} = 51$.

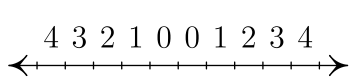
15. **Answer: (E)**

    Note that a_i has exactly $i + 1$ ones and the rest are zeroes. By rules of divisibility by 9, the sum of the digits must be a multiple of 9, so we must have $9 | (i + 1)$, which have a_i divisible by 9.

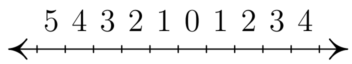
    For divisibility of 11, we claim that $11 | a_i$ if and only if i is odd. Notice that 11, 1001, 100001, ... are all divisible by 11 by the sum of odd powers factorization of 1 and 10. Because a_i for odd i are just linear combinations of these, they are multiples of 11. Because 1 more than a multiple of a multiple of 11 is not a multiple of 11, even i fail on the basis of odd i's success. Combining, we must have $(n + 1) \mid 18$, and the answer is the number of $n = 1, 2, \ldots, 2021$ that are congruent to 17 (mod 18). Thus, the answer is $\lfloor \frac{2021-1}{18} \rfloor = 112$.

16. **Answer: (D)**

    We use an area-based approach along with number lines, where are total area is $10 \cdot 10 = 100$. We can stop worrying about what happens when at least one of a and b is an integer, because those cases are negligible. Below is the number line representing the value of $\lfloor |a| \rfloor$ in the intervals. If seeing what happens on $a \in (n, n + 1)$ is too complicated, plugging in $a = n + \frac{1}{2}$ will suffice.



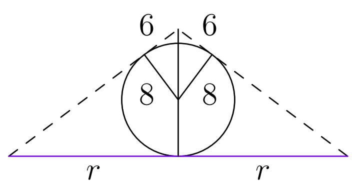
Below is the number line representing the value of $|\lfloor b \rfloor|$ in the intervals.



We seek the number of $10 \cdot 10$ pairs of a and b intervals where $\lfloor |a| \rfloor = |\lfloor b \rfloor|$. Doing simple casework on the equal value, we have $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 0 \cdot 1 = 18$ cases, so our answer is $\frac{18}{10 \cdot 10} = \frac{9}{50}$.

17. **Answer: (D)**

Consider the cross section of the sphere below. The direction of the light is indicated by the dashed lines and the shadow is indicated by the purple segment. The center of the sphere, one of the tangent points, and the lightbulb form a triangle with sides $6, 8$, and $8 + 2 = 10$. This is a right triangle, similar to one of the larger right triangles in the cross section with legs of length r and $8+8+2 = 18$. It follows that $\frac{8}{6} = \frac{r}{18} \implies r = 24$. So the shadow is a circle with radius 24, and area is $24^2 \cdot \pi = 576\pi$.



18. **Answer: (C)**

Putting terms of different degrees on different sides, we can factor to obtain;

$$\begin{cases} 96(y+z) = (x+y)(x+z) \\ 24(x+z) = (y+z)(x+y) \\ 54(x+y) = (x+z)(y+z) \end{cases}$$

Multiplying these and simplifying yields $(x+y)(y+z)(x+z) = 96 \cdot 24 \cdot 54$, and substituting gives

$$96(y+z) = \frac{96 \cdot 24 \cdot 54}{(y+z)} \implies (y+z)^2 = 24 \cdot 54 = 1296 \implies y+z = 36.$$

Similarly, $x+z = 72$ and $x+y = 48$. So $(x, y, z) = (42, 6, 30)$ and $xyz = 42 \cdot 6 \cdot 30 = 7560$, as desired.

19. **Answer: (B)**

Observe that

$$(d_1 - d_2) | d_1 d_2 \implies (k-1)d_2 | k \cdot (d_2)^2 \implies (k-1) | k d_2.$$

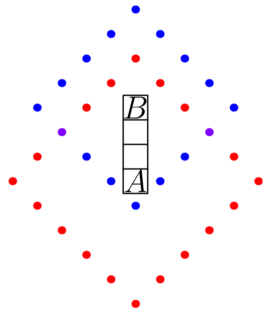
Because $\gcd(k, k-1) = 1$, it follows that $(k-1)|d_2$. Since $d_2|1296$, we must also have $(k-1)|1296$. Note that we cannot have $6|(k-1)$, because this will result in k being neither a multiple of 2 nor 3.

Thus, we need only to check $(k-1)|16$ and $(k-1)|81$. Note that k must be of the form $2^a \cdot 3^b$ for non-negative integers a and b, so our desired answer is $2 + 3 + 4 + 9 = 18$.

20. **Answer: (D)**

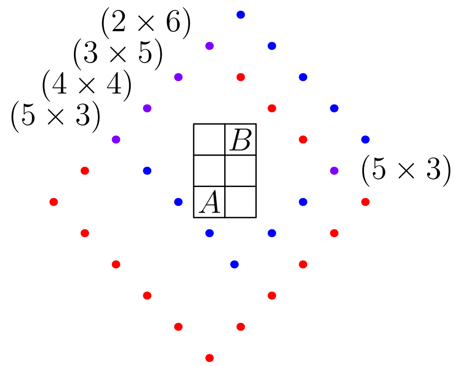
We do cases on whether A to B is three steps in one direction or two steps in one direction and one step in a perpendicular direction.

Case 1. Three steps straight from A to B.



The red dots mark the places of squares that are 5 away from A and the blue dots mark the squares that are 4 away from B. The two purple dots are valid placements for C. However, we can also have A on top and B on bottom, or A and B aligned horizontally, so we have 4 symmetries. There are also $2 \cdot 2$ ways to embed the 4×4 hull of ABC into our 5×5 grid, so our total for this case is $4 \cdot 4 \cdot 2 = 32$.

Case 2. Two steps one direction, one step in a perpendicular direction from A to B.



We can embed the 2×6 hull in 0 ways (we only have a 5×5 grid), a 3×5 or 5×3 hull in 3 ways, and a 4×4 in $2 \cdot 2 = 4$ ways. This gives $1 \cdot 0 + 3 \cdot 3 + 1 \cdot 4 = 13$ ways. However, we must multiply this by 8 to accommodate for the 8 positions (symmetric) of B relative to A, giving $13 \cdot 2^3 = 104$ cases here. Our final answer is $104 + 32 = 136$.

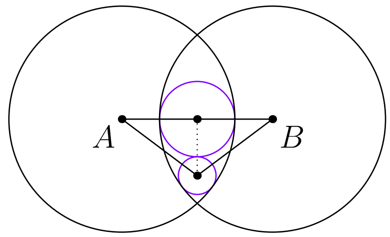
21. **Answer: (C)**

Let A and B be the centers of ω_a and ω_b respectively. Let $R = 1$ be the radius of ω_a and ω_b and $D = \frac{4}{3}$ be the distance between A and B. Note that the centers of ω_1 and ω_2, say O_1 and O_2 respectively, lie on a line that is both perpendicular to AB and equidistant from A and B.

Because $\triangle AO_1O_2 \cong \triangle BO_1O_2$, we have that $\frac{D}{2}$ is the length of the A-altitude of AO_1O_2. We have $AO_1 = R - r_1$, $AO_2 = R - r_2$, and $O_1O_2 = r_1 + r_2$, so $\triangle AO_1O_2$'s perimeter is $2R$. Thus, by Heron's Formula $[\triangle AO_1O_2] = \sqrt{Rr_1r_2(R - r_1 - r_2)} = \frac{1}{2} \cdot \frac{D}{2} \cdot (r_1 + r_2)$. Substituting known values, we have

$$\sqrt{1 \cdot r_1r_2(1 - \tfrac{1}{2})} = \frac{\frac{4}{3}}{4} \cdot \frac{1}{2},$$

whence $r_1 \cdot r_2 = \frac{1}{18}$.



Remark: In this specific case, $\triangle AO_1O_2$ is actually a right triangle with lengths in the ratio $3:4:5$, which is why the diagram has one of the centers lying on AB.

22. **Answer: (C)**

Note that $\sum(r_a+r_b)(r_c+r_d)$ has $15\cdot 4=60$ terms. We can also see that all terms are in the form r_ir_j or $(r_i)^2$. The only way $(r_i)^2$ are produced is $(r_i+r_j)(r_i+r_k)$ and there are $\binom{3}{2}=3$ pairs of these for a given value of i.

So three copies of each $(r_i)^2$ is produced, and there are $\frac{60-3\cdot 4}{6}=8$ copies of r_ir_j by symmetry. By Vieta's, our desired answer is equal to

$$3\sum(r_i)^2+8\sum r_ir_j=42.$$

23. **Answer: (A)**

The crux of this problem is noting that the anteater could only make forward progress. Let e_i be the life expectancy of the ant where the closest path from the anteater to the ant is a distance of i edges, and $e_0=0$. Note that $e_1=\frac{1}{3}\cdot -\frac{1}{2}+\frac{2}{3}e_1+1 \implies e_1=\frac{5}{2}$. Here, $-\frac{1}{2}$ comes from the death of the ant occurring before the step has gone through completely. Now, for e_2, the anteater chooses one of two congruent paths. We have $e_2=\frac{1}{3}\cdot 0+\frac{2}{3}e_2+1 \implies e_2=3$. Similarly, we obtain $e_3=\frac{2}{3}e_1+\frac{1}{3}e_3+1 \implies e_3=4$. The life expectancy of the ant is $\frac{1}{8}e_0+\frac{3}{8}e_1+\frac{3}{8}e_2+\frac{1}{8}e_3=\frac{41}{16}$.

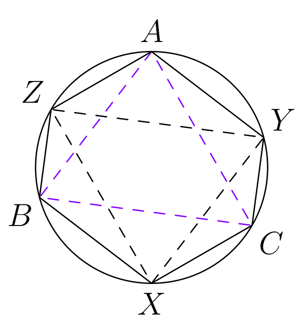
24. **Answer: (D)**

Note that $[BCYZ]+[CAZX]+[ABXY]$ is equivalent to counting the central hexagon three times, the smaller triangles sharing a side with that hexagon twice, and the outer triangles just once. If H is the orthocenter of triangle ABC, it is well-known that H and X are reflections of each other across the midpoint of BC. Therefore, $\triangle BHC$ and $\triangle CXB$ are congruent.

Using analogous reasoning and summing congruent areas, we see that $[\triangle ABC]=\frac{[AZBXCY]}{2}=1$. Similarly, $[\triangle XYZ]=1$. Notice that $[\triangle ABC]+[\triangle XYZ]$ counts the central hexagon two times and the smaller triangles sharing a side with that hexagon once. Therefore, adding in one copy of $[AZBXCY]$ will equate to the sum of all three rectangles;

$$[AZBXCY]+[\triangle ABC]+[\triangle XYZ]=4=[BCYZ]+[CAZX]+[ABXY],$$

so we have $[CAZX]+[ABXY]=3$, as desired.



Solution 2 (Circumcenter)

Let O denote the circumcenter. Since $[BCYZ] = 1$, we have $[\triangle BOZ] = [\triangle COB] = \frac{1}{4}[BCYZ] = \frac{1}{4}$. Observe that $[\triangle AOC] = [\triangle AOZ], [\triangle AOB] = [\triangle BOX]$, implying that the hexagon's area equals

$$2([\triangle AOZ] + [\triangle BOX] + [\triangle BOZ]) = 2\left([\triangle AOC] + [\triangle AOB] + \frac{1}{4}\right)$$

In addition, $[ABC] = [AOC] + [AOB] + [BOC] = [AOC] + [AOB] + \frac{1}{4}$. Hence, the sum of the areas is equal to $3([AOC] + [AOB] + \frac{1}{4}) = 3$, so it follows that $[\triangle AOC] + [\triangle AOB] = \frac{3}{4}$.

Since $[XZAC] = 4[AOC], [YXBA] = 4[AOB]$, $[XZAC] + [YXBA] = 4([\triangle AOC] + [\triangle AOB]) = 3$.

25. **Answer: (D)**

The problem asks for when $2^{2b} + 11^{2b}$ and $2^a - 3^b$ have the same number of powers of 5.

First, when b is even, the numerator $4^b + 121^b$ is not divisible by 5, so the denominator must also not be divisible by 5. So $2^a \not\equiv (2^3)^b \pmod 5$ if and only if $a \not\equiv 3b \pmod 4$. This gives $75 \cdot 5 = 375$ solutions.

When b is odd, $4^b + 121^b$ has at least 3 powers of 5 due to the sum of b-th powers factorization for odd b. In order for $2^a \equiv 3^b \pmod{125}$, we must have $2^a \equiv (2^7)^b \pmod{125}$. Note that $\text{ord}_2(125) = 100$ (taking 2^{10} and/or 2^{20} modulo 125 will work). Therefore, $7b \equiv a \pmod{100}$. It is clear that $7b = a$ must be satisfied. To verify that all (a, b) with $a = 7b$ work, we check with Lift-the-Exponent Lemma;

$$v_5(121^b + 4^b) = v_5(125) + v_5(b) = v_5(128^b - 3^b)$$

where $v_5(x)$ is the maximum possible k such that 5^k divides x. So when b is odd, there are 5 solutions. Thus, we have in total, 380 solutions.