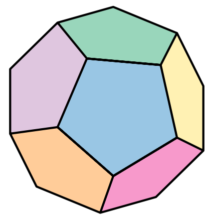
November ARML



Credits

This contest was created with the help of many people over months of work. The MAC would like to thank the following for their contributions to the contest:

- Problem writers: dchenmathcounts, skyscraper, youyanli, Depsilon0, and vvluo, who wrote the problems that appeared on the final contest.
- Problem proposers: innumerateguy, freeman66, vvluo, Radio2, and smartatmath, who proposed problems to the contest.
- Test-solvers: nukelauncher, Frestho, mathgirl199, and djmathman, who gave comments, suggestions, and advice for the contest.

Rules

The rules, procedures, and format of this mock ARML follow.

- The Individual Rounds consist of five rounds, each with two questions. One point is awarded for each question answered correctly for a total of 10 points possible per person.
- Each round is 10 minutes long. In this mock contest, breaks between rounds are permitted, though not encouraged.
- Answers should be sent via AoPS PM to dchenmathcounts, skyscraper, innumerateguy, youyanli, and vvluo.
- The only acceptable aids are a pencil or pen, an eraser, blank paper, a straightedge, and a compass. In particular, graph paper and protractors are not permitted.
- Calculators will not be allowed on any part of the ARML contest.
- During the Individual round there is to be absolutely no communication between students once the questions are handed out. This includes communication between students who have already finished the contest. The only exception is in the official discussion forum set up by the MAC.

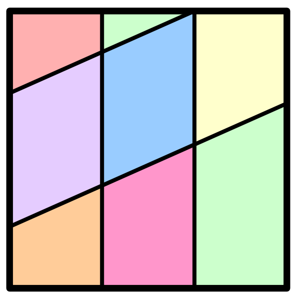
Problems 1 and 2

Problem 1

Suppose $ABCD$ is a rectangle with $AB = 14$ and $BC = 28$. If point P outside $ABCD$ satisfies the conditions $PA = 13$ and $PB = 15$, compute the length of PC.

Problem 2

The numbers $\{1, 2, \ldots, 8\}$ are placed in each of the cells in the magic square below such that the number in each cell is distinct, and the sum of all numbers in each slanted row and each column is the same. Compute the number of ways that the magic square can be filled out.



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Problems 3 and 4

Problem 3

Determine all values of a such that the equation

$$ax^2 - (a+4)x + \frac{9}{2} = 0$$

only has one real solution over x.

Problem 4

Compute the smallest positive integer n such that $9(n+3)$ divides $4n! + n + 5$.

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Problems 5 and 6

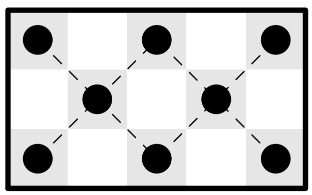
Problem 5

Let $a_1, a_2, a_3, \ldots$ be a sequence that satisfies $a_1 = a_2 = 1$ and $4a_n = 9a_{n-2} - a_{n-1}$. Compute

$$\sum_{n=1}^{\infty} a_n \cdot \left(\frac{2}{3}\right)^n .$$

Problem 6

Consider a 3×5 rectangle colored in a checkerboard pattern, with its corner squares being black. Rocks of different colors are put on each black square. A valid move consists of taking a stack of rocks and placing it over a diagonally adjacent square with at least one rock on it. If all of the rocks end up in the same stack, how many ways can the rocks in the stack be ordered?



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Problems 7 and 8

Problem 7

Consider $\triangle ABC$ with a right angle at B and $AB = 2$. Let point $D \neq C$ lie on segment AC with $AD = 4$, and let E be a point such that segment DE is a diameter of the circumcircle of $\triangle BDC$. If $AE = 10$, compute AC.

Problem 8

The mad scientist Kyouma is traveling on a number line from 1 to 2020, subject to the following rules:

- He starts at 1.
- Each move, he randomly and uniformly picks a number greater than his current number to go to.
- If he reaches 2020, he is instantly teleported back to 1.
- There is a time machine on 199.
- A foreign government is waiting to ambush him on 1729.

What is the probability that he gets to the time machine before being ambushed?

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Problems 9 and 10

Problem 9

Let $\mathcal{H}(x)$ be a function on the positive integers such that

- $\mathcal{H}(1) = 1$, and
- for integers $n > 1$, $2\mathcal{H}(n) = \sum_{i|n} \mathcal{H}(i)$.

Compute the smallest positive integer $n > 1$ that satisfies $\mathcal{H}(n) = n$.

Problem 10

Three mutually tangent spheres with radii of 1 are tangent to a table, and a cone is tangent to all three spheres with its tip oriented towards the table. If the cone has height $\sqrt{2}$ and its tip is $\frac{3+\sqrt{3}}{3}$ units above the table, compute the radius of the cone.

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