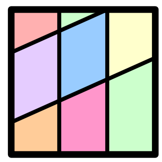
HMMT November Mock

General Round

1. **[2]** Suppose $ABCD$ is a rectangle with $AB = 14$ and $BC = 28$. If point P outside $ABCD$ satisfies the conditions $PA = 13$ and $PB = 15$, determine the length of segment PC.

2. **[2]** Determine all values of a such that the equation $ax^2 - (a+4)x + \frac{9}{2} = 0$ only has one solution x.

3. **[3]** Find the number of ways to place $\{1, 2, \ldots, 8\}$ in each of the cells of the *slanted* magic square below such that each cell is distinct, and the sum of all numbers in each row and column is the same.



4. **[4]** Find the smallest positive integer n such that $9(n+3)$ divides $4 \cdot n! + n + 5$.

5. **[5]** Let $a_1, a_2, a_3, \ldots$ be a sequence that satisfies $a_1 = a_2 = 1$ and $4a_n = 9a_{n-2} - a_{n-1}$. Compute

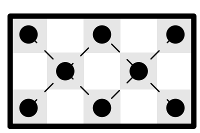
$$\sum_{n=1}^{\infty} a_n \cdot \left(\frac{2}{3}\right)^n.$$

6. **[6]** Consider $\triangle ABC$ with a right angle B and $AB = 2$. Point $D \neq C$ lies on segment $\overline{AC}$ with $AD = 4$, and E exists so that $\overline{DE}$ is a diameter of the circumcircle of $\triangle BDC$. If $AE = 10$, compute AC.

7. **[6]** Find the smallest positive integer $n > 1$ that satisfies $\mathcal{H}(n) = n$ given that $\mathcal{H}(x)$ is a function on positive integers satisfying:

   - $\mathcal{H}(1) = 1$, and
   - For integers $n > 1$, $2\mathcal{H}(n) = \sum_{i|n} \mathcal{H}(i)$.

8. **[7]** Rocks of different colors are put on each black square of the below 3×5 checkerboard. In a move, a stack of rocks from a square are placed over a diagonally adjacent square with at least one rock on it. All of the rocks end up in the same stack. How many ways can the rocks in the stack be ordered?



9. **[7]** Three mutually tangent spheres with radii of 1 are tangent to a table, and a cone $\mathcal{C}$ is tangent to all three spheres with its tip oriented towards the table. If $\mathcal{C}$ has height $\sqrt{2}$ with its tip lying $\frac{3+\sqrt{3}}{3}$ units above the table, determine the radius of $\mathcal{C}$.

10. **[8]** The *mad* scientist Kyouma is traveling on a number line labelled $1, 2, 3 \cdots 2020$. He randomly picks a number greater than his current number to go to for each move and teleports to 1 after reaching point 2020. A time machine awaits on 199 and a foreign government is ready to ambush him on 1729. What is the probability that he gets to the time machine before being ambushed?