Solutions to the JMC 10

Mathematical Advancement Committee

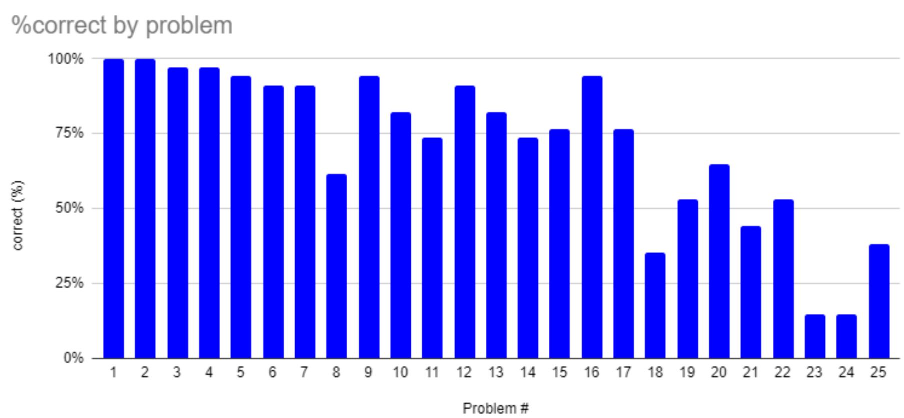
July 28th, 2020

Answer Key

CEEDB CDAED BDBCE ECEEB CDDDA

Statistics

The leaderboard can be found on our main AoPs thread. There were a total of **40** submissions and the mean score was $114.\overline{27}$. Question 25 was the most favorited question among users. Here are the statistics:



Once again, thank you all for making this a successful mock!

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§ 1 Problem (skyscraper)

What is the value of x satisfying $4^3 = \sqrt{x^4\sqrt{4^x}}$?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 12

§ 1.1 Solution

We may simplify as $\sqrt{x^4\sqrt{4^x}} = x^2 \cdot 2^{\frac{x}{2}} = 2^6$. Suppose that $x = 2^k$. It follows $x^2 \cdot 2^{\frac{x}{2}} = 2^{2k} \cdot 2^{2^{k-1}} = 2^6 \Rightarrow 2k + 2^{k-1} = 6 \Rightarrow k = 2 \Rightarrow x = 2^2 = 4$, corresponding to answer choice C.

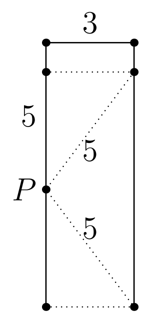
§ 2 Problem (innumerateguy/skyscraper)

Maya starts at a point and walks 5 meters north, 3 meters east, and n meters south. If Maya is now 5 meters from her original location, what is the sum of all possible n?

(A) 0 **(B)** 1 **(C)** 4 **(D)** 9 **(E)** 10

§ 2.1 Solution

The key is to notice that $3-4-5$ right triangles are involved here, where 5 (the distance from her new point to her original point) is the hypotenuse. Assume Maya starts at point P in the following diagram.



Thus Maya can move 1 or $1+4+4=9$ meters south. The answer is $1+9=10$, or answer choice E.

§ 3 Problem (skyscraper)

Nonnegative integers a and b satisfy $3^a \cdot 4^b = 6^6 \cdot 9^b$. What is $a + b$?

(A) 4 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 15

§ 3.1 Solution

Prime factorize $6^6 \cdot 9^b$ as $2^6 \cdot 3^{6+2b}$ and $3^a \cdot 4^b$ as $2^{2b} \cdot 3^a$. So $2b = 6 \Rightarrow b = 3$ and $a = 6 + 2b = 12$. Hence the answer is $a + b = 3 + 12 = 15$, corresponding to answer choice E.

§ 4 Problem (skyscraper)

What is the sum of the digits of the least 3 digit integer divisor of $9! = 9 \cdot 8 \cdot 7 \cdots 1$?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

§ 4.1 Solution

Note $9! = 9 \cdot 8 \cdot 7 \cdots 1 = 2^8 \cdot 3^4 \cdot 5 \cdot 7$. Our divisor cannot have other primes as divisors or a power of that prime exceeding the one in the prime factorization of 9!. Working from 100, our answer is $3 \cdot 5 \cdot 7 = 105$, corresponding to answer choice D.

§ 5 Problem (dchenmathcounts)

Five years ago, the average of Albert and Bessy's ages was the same as Bessy's current age. If Albert is 20 years old, how old is Bessy?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 30

§ 5.1 Solution

Say Albert's age is a and Bessy's age is b. Then $\frac{(a-5)+(b-5)}{2} = b$, implying $a + b - 10 = 2b$, or $a - 10 = b$. Thus Bessy is 10 years old, corresponding to answer choice B.

§ 6 Problem (dchenmathcounts)

Cars A and B, travelling at constant, different speeds, are headed directly from Austin to Boston and from Boston to Austin respectively. Car A leaves at 9:00 AM, and Car B leaves an hour later. If the two cars meet when Car A is $\frac{2}{3}$ of the way to Boston, and Car A arrives at Boston at 3:00 PM, when does Car B reach Austin?

(A) 6:00 PM **(B)** 6:30 PM **(C)** 7:00 PM **(D)** 7:30 PM **(E)** 8:00 PM

§ 6.1 Solution

Note that it takes Car A six hours to get to Boston, so the cars meet at 1:00 PM. Since Car B leaves during 10:00 AM, it takes Car B three hours to travel $\frac{1}{3}$ of the distance. Thus it will take Car B nine hours total, so it will reach Austin at 7:00 PM, corresponding to answer choice C.

§ 7 Problem (skyscraper)

Three kids stand in a row from left to right. Simultaneously, each kid randomly points left or right. What is the probability that no two adjacent kids point at each other?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

§ 7.1 Solution

Call the kids from left to right a_1, a_2, and a_3. We proceed to complementary count. Assume a_1 and a_2 point to each other, there are two cases as a_3 can point left of right with interfering. The same applies for a_2 and a_3. There is no overlap, so the answer is $1 - \frac{2 \cdot 2}{2^3} = \frac{1}{2}$, corresponding to answer choice D.

§ 8 Problem (skyscraper/innumerateguy)

The operator $\tau(n)$ denotes the number of divisors of n. What is the sum of the digits of the smallest positive integer n that satisfies $\tau(\tau(n^2)) = 4$?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

§ 8.1 Solution

Note that $\tau(\tau(n^2)) = 4 \implies \tau(n^2) = p_1^3$ or $x = p_1 p_2$. Also note that $\tau(n^2) = 1 \pmod 2$, so none of p_1, p_2 can be 2. For the first case, the smallest p_1 is 3 giving $\tau(n^2) = 27$. This gives a smallest n of $2 \cdot 3 \cdot 5 = 30$. Clearly, this is the smallest n for this case.

For the second case, the smallest p_1, p_2 are $p_1 = 3$ and $p_3 = 5$. This gives $\tau(n^2) = 15$ and the smallest n is $2^2 \cdot 3 = 12$. Clearly, this is the smallest n for this case.

The answer is $1 + 2 = 3$, corresponding to answer choice A.

§ 9 Problem (skyscraper)

If γ is a root of $x^2 + x + 1$, what is the value of $(1 - \gamma)(\gamma + 2)$?

(A) -3 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 3

§ 9.1 Solution

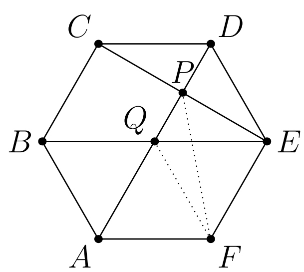
Note $(1 - \gamma)(\gamma + 2) = -\gamma^2 - \gamma + 2 = -(\gamma^2 + \gamma + 1) + 3 = 3$, as γ is a root of $x^2 + x + 1$. Answer choice E.

§ 10 Problem (skyscraper)

In regular hexagon $ABCDEF$ with sides of length 3, diagonals $\overline{AD}, \overline{BE}$, and $\overline{CE}$ intersect at two points P and Q in the hexagon's interior. Find the area of $\triangle FPQ$.

(A) $\frac{3\sqrt{3}}{8}$ **(B)** $\frac{3\sqrt{3}}{4}$ **(C)** 2 **(D)** $\frac{9\sqrt{3}}{8}$ **(E)** $\frac{9\sqrt{3}}{4}$

§ 10.1 Solution



Note that $\overline{PQ}$ has length $\frac{3}{2}$ and the height of $\triangle FPQ$ is $3 \cdot \frac{1}{2} \cdot \sqrt{3} = \frac{3\sqrt{3}}{2}$. By the area formula, the area is $\frac{9\sqrt{3}}{8}$, corresponding to choice D.

§ 11 Problem (skyscraper)

Jack has eight sticks of different lengths in the set $\{1, 2, 3, \dots, 7, 8\}$. How many nonempty subsets of these eight sticks can Jack choose so the range of the lengths is at most 4 meters?

(A) 64 **(B)** 79 **(C)** 80 **(D)** 81 **(E)** 128

§ 11.1 Solution

Say that the range is r meters. Then there are $8 - r$ ways to pick the length of the shortest stick, as it uniquely determines the longer stick, and the valid lengths of the shorter stick are $\{1, 2, \dots, 8 - r\}$.

If the range is r meters, then there are $r - 1$ sticks in the range that can either be included or excluded from the set, so we multiply by 2^{r-1}. So the number of subsets Jack can choose is

$$\sum_{r=1}^{4} (8 - r) 2^{r-1} = 79,$$

corresponding to answer choice B.

§ 12 Problem (skyscraper)

Alice, Albert, Bella, Bernie, Carol, and Carl are randomly split into two indistinguishable groups of three. What is the probability that no two people whose names start with the same letter are in the same group?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

§ 12.1 Solution

There are $\binom{6}{3} = 20$ ways to choose the groups distinguishably, so there are half that many, 10 ways to choose the groups indistinguishably, since each group split is counted twice in 6 choose 3.

There must be one A-person(name starts with A), one B-person, and one C-person in each group. Suppose Alice is in a particular group. There are 2 different ways to choose which B-person shares a group with her. Then there are 2 ways to choose which C-person shares a group with her. This uniquely determines an indistinguishable grouping. Thus, the fraction of cases that work is $\frac{4}{10} = \frac{2}{5}$, corresponding to choice D.

§ 13 Problem (skyscraper)

Triangle ABC exists in the coordinate plane such that $\overline{AB}, \overline{AC}$, and $\overline{BC}$ have slopes equal to $1, \frac{1}{3}$ and -1 respectively. If $\overline{AB}$ has length $\frac{8}{3}$, what is the length of $\overline{BC}$?

(A) $\frac{2\sqrt{2}}{3}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{4\sqrt{2}}{3}$ **(D)** 4 **(E)** $\frac{8\sqrt{2}}{3}$

§ 13.1 Solution

Without loss of generality, let $A = (0, 0)$. Then $B = (\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3})$. The line through B with slope -1 is $x + y = \frac{8\sqrt{2}}{3}$ and the line through A with slope $\frac{1}{3}$ is $y = \frac{1}{3}x$, implying $\frac{4}{3}x = \frac{8\sqrt{2}}{3} \Rightarrow x = 2\sqrt{2}$. The difference between the x coordinates of B and C is $\frac{2\sqrt{2}}{3}$, implying $BC = \frac{4}{3}$, corresponding to answer choice B.

§ 14 Problem (skyscraper)

Steven has 4 lit candles and each candle is blown out with a probability $\frac{1}{2}$. After he finishes blowing, he randomly selects a possibly empty subset out of all the candles. What is the probability his subset has at least one lit candle?

(A) $\dfrac{5}{16}$ **(B)** $\dfrac{1}{2}$ **(C)** $\dfrac{175}{256}$ **(D)** $\dfrac{11}{16}$ **(E)** $\dfrac{3}{4}$

§ 14.1 Solution

This is basically equivalent to choosing an ordered pair of subsets (A, B) and finding the probability that $A \cap B \neq \emptyset$. A here represents the subset of lit candles and B represents the subset that Steven chooses afterwards. We can do complementary counting and find the probability that $A \cap B = \emptyset$. For each of the candles, there are 3 choices: in A, in B, or in neither. So, there are $3^4 = 81$ possible ordered (A, B) such that $A \cap B = \emptyset$. There are $(2^4)^2 = 256$ pairs (A, B) in total so there are 175 pairs (A, B) such that $A \cap B \neq \emptyset$.

Our probability is $\frac{175}{256}$ corresponding to answer choice C.

§ 15 Problem (depsilon0)

What is the sum of all positive integers b greater than 1 such that the base 10 numbers 13, 167, and 233 all have the same last digit when expressed in base b?

(A) 2 **(B)** 13 **(C)** 17 **(D)** 22 **(E)** 35

§ 15.1 Solution

Suppose that 13, 167, and 233 have all have the digit d base b. Then

$$\begin{cases} 13 \equiv d \pmod{b} \\ 167 \equiv d \pmod{b} \\ 233 \equiv d \pmod{b} \end{cases}$$

Subtracting the congruences, we get

$$\begin{cases} 66 \equiv 0 \pmod{b} \\ 154 \equiv 0 \pmod{b} \\ 220 \equiv 0 \pmod{b} \end{cases}$$

So $b|154$, $b|66$, and $b|220$. Since $\gcd(154, 66, 220) = 22$, we must have that $b|22$. Since $b \neq 1$, the only possibilities for b are $2, 11, 22$, which have a sum of 35, corresponding to choice E.

§ 16 Problem (freeman66)

What is the value of

$$\frac{(2019+2020)(2020+2021)(2021+2019)+2019\cdot 2020\cdot 2021}{2019\cdot 2020+2020\cdot 2021+2021\cdot 2019}?$$

(A) 2019 **(B)** 2020 **(C)** 2021 **(D)** 3030 **(E)** 6060

§ 16.1 Solution

For the sake of simplicity, let $x = 2020$, implying that $x-1 = 2019$ and $x+1 = 2021$. We can simplify the monstrous expression in terms of x as:

$$\frac{(x+x-1)(x+x+1)(x+1+x-1)+x(x-1)(x+1)}{(x-1)x+x(x+1)+(x+1)(x-1)}$$

Simplifying further gives:

$$\frac{(2x-1)(2x+1)(2x)+x(x-1)(x+1)}{x^2-x+x^2+x+x^2-1}$$

$$=\frac{8x^3-2x+x^3-x}{3x^2-1}$$

$$=\frac{9x^3-3x}{3x^2-1}$$

$$=3x$$

Our answer is 6060, corresponding to answer choice E.

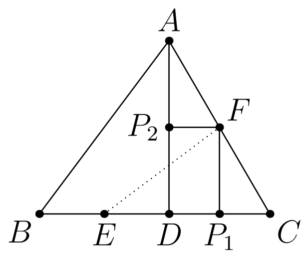
§ 17 Problem (starchan)

Let triangle ABC be acute and point D be the altitude from point A to $\overline{BC}$. If $AD = 24$ and $BC = 32$, what is the distance between the midpoints of $\overline{BD}$ and $\overline{AC}$?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 25

§ 17.1 Solution

Triangle FP_1C is similar to triangle ADC with a scale factor of two (because of equal angles), so it follows that $\overline{FP_1} = \frac{AD}{2} = \frac{24}{2} = 12$. It also follows that $\overline{P_1C} = \overline{P_1D}$ so the length $EP_1 = \frac{BC}{2} = \frac{32}{2} = 16$ since $EP_1 = ED + DP_1$ and $BE = ED, DP_1 = P_1C$. Segment $\overline{EF}$ is the hypotenuse of a right triangle with legs of lengths 12 and 16, so we have $\overline{EF} = 20$ by the Pythagorean theorem, corresponding to answer choice C.



§ 18 Problem (innumerateguy)

Harvey *transforms* a 4-digit number by reversing the order of its digits, subtracting 1 from all digits that are 1 more than a multiple of 3, and adding 1 to all even digits in that order. Harvey obtains 7793 after transformation. How many distinct numbers could he have transformed?

(A) 4 **(B)** 6 **(C)** 12 **(D)** 16 **(E)** 24

§ 18.1 Solution

Let the transformations in order be f, g, h. Then, we want to find the number of x such that $h(g(f(x))) = 7793$. We can essentially ignore the first transformation because each x such that $h(g(x)) = 7793$ bijects to some y such that $h(g(f(y))) = 7793$. Now, we note that each digit is essentially independent of each other. So, we can do casework on the digits. For 7, we start with either $6, 7$. For 9, we can either start with $8, 9$. For 3, we can either start with $2, 3, 4$. So, we have $2^3 \cdot 3 = 24$ original numbers, corresponding to choice E.

§ 19 Problem (youyanli)

Circle ω_A has center M and radius 5. Line ℓ intersects ω_A at A, B such that $AB = 7$. Point P is on line ℓ outside of ω_A with $PA = 9$ and P closer to A. Circle ω_B with diameter $\overline{PM}$ intersects ω_A at two points X and Y. If the length of $\overline{XY}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m + n$?

(A) 23 **(B)** 29 **(C)** 41 **(D)** 73 **(E)** 133

§ 19.1 Solution

Note that the power of P with respect to ω_A is equal to $PA \cdot PB$ which is also equal to $PM^2 - r_A^2$ where $r_A = 5$ is the radius of ω_A.

Thus,

$$PM^2 - 5^2 = PA \cdot PB = 9 \cdot 16 = 144.$$

Now, $PM^2 = 144 + 25 = 169$, so it follows that $PM = 13$.

Note that $\angle PXM = \angle PYM = 90°$ by Thale's Theorem.

Since X and Y are symmetric across PM (the line throught the center of the two circles), we have that $XY \perp PM$, so XY is 2 times the X-altitude in $5 - 12 - 13$ right triangle PXM.

This altitude has length $\frac{5 \cdot 12}{13} = \frac{60}{13}$, so twice that is $XY = \frac{120}{13}$,giving 133 and corresponding to answer choice E.

§ 20 Problem (skyscraper)

Consider triangle ABC with medians $\overline{BE}$ and $\overline{CF}$ that intersect at G. If $AG = BC = 8$ and $CG = 6$, what is the length of $\overline{GE}$?

(A) $\sqrt{3}$ **(B)** $\sqrt{7}$ **(C)** $2\sqrt{2}$ **(D)** $2\sqrt{7}$ **(E)** $4\sqrt{2}$

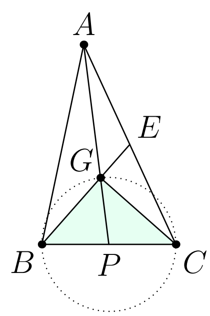
§ 20.1 Solution

The key step is to realize that $\angle BGC$ is right. We can prove this in two ways.

The first way is let $\overline{AG}$ meet $\overline{BC}$ at point P. The length of $\overline{PG}$ is $\frac{8}{2} = 4$. So $BP = PG = PC = 4$ and $\triangle BGC$ can be circumscribed in a circle with diameter $\overline{BC}$. It follows that $\angle BGC = 90°$.

The second way is to extend point P to meet $\overline{GC}$ at midpoint M. Since $\triangle GPC$ is isosceles, $CM = GM$. We also have $BP = PC$ and $\angle PMC = 90°$. $\triangle PMC \sim \triangle BGC$ by side-angle-side and the rest follows.

By the Pythagorean theorem, $\overline{BG}$ has length $2\sqrt{7} \Rightarrow GE = \frac{2\sqrt{7}}{2} = \sqrt{7}$, corresponding to choice B.



§ 21 Problem (dchenmathcounts)

What is the base-10 sum of all positive integers such that, when expressed in binary, have 7 digits and have no two consecutive 1's?

(A) 667 **(B)** 921 **(C)** 963 **(D)** 1022 **(E)** 1411

§ 21.1 Solution

We count the number of times each digit can be 1. Note the first two digits have to be 10 for obvious reasons. So the number of times the first digit can be 1 is just the number of 7 digit numbers that satisfy the condition.

Let $F(n)$ indicate the number of n digit sequences, with leading 0 allowed, that satisfies the condition. Then we want to find $F(5)$. Note $F(1) = 2$ and $F(2) = 3$. Also note that $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$, based on casework whether the last digit is a 1 or not. (If the last digit is a 1, the last two digits are 01, so you can remove them to create a recursion. Otherwise you can just remove the last digit of 0 to create a recursion.) Thus

$$F(3) = F(2) + F(1) = 3 + 2 = 5$$
$$F(4) = F(3) + F(2) = 5 + 3 = 8$$
$$F(5) = F(4) + F(3) = 8 + 5 = 13.$$

Now let's count the number of times the third, fourth, fifth, sixth, and seventh digits can be 1. Note the third and seventh digit are symmetric and the fourth and sixth digit are symmetric.

If the third digit is 1 then the fourth digit is 0. Thus the number of valid sequences is just $F(3) = 5$.

If the fourth digit is 1 then the third and fifth digits are 0. Thus the number of valid sequences is just $F(2) = 3$.

If the fifth digit is 1 then the fourth and sixth digits are 0, and the third and seventh digits can be chosen at will. Thus the number of valid sequences is $2^2 = 4$.

Thus the total sum is

$$2^6 \cdot 13 + 2^4 \cdot 5 + 2^3 \cdot 3 + 2^2 \cdot 4 + 2^1 \cdot 3 + 2^0 \cdot 5 = 963,$$

which corresponds to answer choice C.

§ 22 Problem (dchenmathcounts)

What is the units digit of the remainder when $17^7 + 17^2 + 1$ is divided by 307^2?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

§ 22.1 Solution

The crucial observation is that $17^2 + 17 + 1 = 307$. The problem is quite easy from here.

Substitute $17 = x$. Then we want to find the remainder when $x^7 + x^2 + 1$ is divided by $(x^2 + x + 1)^2$. Notice that $x^7 + x^2 + 1$ is divisible by $x^2 + x + 1$. (This can be motivated by Roots of Unity.) Dividing gives

$$\frac{x^7 + x^2 + 1}{x^2 + x + 1} = 1 + \frac{x^7 - x}{x^2 + x + 1} = 1 + \frac{x(x^3 - 1)(x^3 + 1)}{\frac{x^3 - 1}{x - 1}} = x(x - 1)(x^3 + 1) + 1.$$

Now we find the remainder of $x(x - 1)(x^3 + 1) + 1$ divided by $x^2 + x + 1$ again. Note

$$x(x - 1)(x^3 + 1) + 1 \equiv x(x - 1)(2) + 1 \equiv 2x^2 - 2x + 1 \equiv -4x - 1 \pmod{x^2 + x + 1}.$$

Now we can substitute $x = 17$ again. Note

$$-4 \cdot 17 - 1 \equiv 307 - 4 \cdot 17 - 1 \equiv 238 \pmod{307},$$

so

$$\frac{17^7 + 17^2 + 1}{307} \cdot 307 \equiv 238 \cdot 307 \pmod{307 \cdot 307}.$$

Note that $238 \cdot 307 \equiv 8 \cdot 7 \equiv 6 \pmod{10}$, so the answer is D.

§ 23 Problem (innumerateguy)

For each positive integer n, let $s(n)$ denote the sum of the digits of n when written in base 10. An integer x is said to be tasty if

$$s(11x) = s(101x) = s(1001x) = \cdots = 18.$$

How many *tasty* integers are less than 10^8?

(A) 6435 **(B)** 8008 **(C)** 10000 **(D)** 11440 **(E)** 12870

§ 23.1 Solution

The first and most critical step is to notice that x is tasty if and only if $s(x) = 9$. We prove the if direction first. If $s(x) = 9$, then $s(10^k + 1) = 9 + 9 = 18$ for $k \geq 1$ as it is impossible for there to be any carrying. If there was a carry, two digits of x sum to over 9, contradicting $s(x) = 9$.

We prove the only if direction now. If x is tasty, consider $s((10^k + 1)x) = 18$ where k is greater than the number of digits in x. Then, there can't be any carrying when adding $10^k \cdot x$ and x. So, $s((10^k + 1)x) = 2s(x) \implies s(x) = 9$.

Now, we want to find the number of tasty integers with 8 or less digits. Consider a sequence of 8 digits $d_1 d_2 \cdots d_8$, allowing leading 0's and $d_1 + d_2 \cdots + d_8 = 9$. [1] Any tasty integer with 8 or less digits corresponds to a unique sequence. We can just add leading 0's to a tasty integer until it has 8 digits and remove leading 0's from a sequence. [2]

By Stars and Bars, there are $\binom{16}{9} = 11140$ sequences, corresponding to answer choice D.

[1] These are all digits because $d_k \leq 9$.

[2] Note that the sequence can't be all 0's, so removing leading 0's will result in a positive integer.

§ 24 Problem (youyanli)

Define $f(k)$ as the number of real solutions x to $x^2 - \lfloor kx \rfloor = 0$. What is the value of

$$f(1) + f(2) + f(3) + \cdots + f(25)?$$

(A) 73 **(B)** 74 **(C)** 75 **(D)** 98 **(E)** 100

§ 24.1 Solution

Note that for $k = 1$, we have by $x^2 \le kx$, so $x^2 \le x$ and $x = 0, 1$. Both solutions work.

We claim that for every $k \ge 2$, we have exactly 4 solutions. Specifically, we claim the solutions are

$$x = 0, \sqrt{k^2 - 2}, \sqrt{k^2 - 1}, k.$$

Note that since the right hand side of $x^2 = \lfloor kx \rfloor$ is an integer, all solutions must be in the form of $\sqrt{t}$ for some positive integer t.

Keeping in mind that x^2 is always an integer, we expand the floor function to get $kx - 1 < x^2 \le kx$.

Thus, we have $x^2 \le kx$ or $x(x - k) \le 0$, which implies $0 \le x \le k$.

Then, we have $kx - 1 < x^2$, or $x^2 - kx + 1 > 0$. The quadratic formula tells us that the roots of $x^2 - kx + 1$ are

$$\frac{k \pm \sqrt{k^2 - 4}}{2},$$

so either $x < \frac{k - \sqrt{k^2 - 4}}{2}$ or $x > \frac{k + \sqrt{k^2 - 4}}{2}$.

Thus, either $0 \le x < \frac{k - \sqrt{k^2 - 4}}{2}$ or $\frac{k + \sqrt{k^2 - 4}}{2} < x \le k$.

There are a total of 98 solutions, corresponding to answer choice D.

Remark: There is a much less rigorous solution that can be seen by graphing the two sides as functions. One can use engineer's induction carefully to get the answer. However, the $\sqrt{k^2 - 2}$ solution is indeed a bit hard to see.

§ 24.2 Solution 2

In all of $f(1), f(2), ...f(25)$, k is positive, so $x \ge 0$, as if x was negative then $x^2 - \lfloor kx \rfloor$ would be positive. Also, x must be the square root of a nonnegative integer, as the right hand side of $x^2 = \lfloor kx \rfloor$ is a nonnegative integer. For $k = 1$ we have that $x^2 = \lfloor x \rfloor$. Clearly if $x \ge \sqrt{2}$ we have $x^2 > \lfloor x \rfloor$ (can easily be shown with induction), so x must be 0 and/or 1, both of which work, meaning that $f(1) = 2$.

Next, we consider when $k \ge 2$. We need to have $x = \sqrt{\lfloor kx \rfloor}$. Suppose we have $\frac{1}{k}(n) \le x < \frac{1}{k}(n+1)$ for some nonnegative integer n. This means that $\lfloor kx \rfloor = n$, or $x = \sqrt{n}$. Plugging this back into $\frac{1}{k}(n) \le x < \frac{1}{k}(n+1)$ gives $\frac{1}{k}(n) \le \sqrt{n} < \frac{1}{k}(n+1)$.

Now multiply the inequality by k and square everything; this gives $n^2 \le k^2 n < n^2 + 2n + 1$. We consider $n^2 \le k^2 n$, which gives $0 \le n \le k^2$. Now we deal with $k^2 n < n^2 + 2n + 1$. This is the same as $n^2 + (2 - k^2)n + 1 > 0$, which is the same as $n(n - (k^2 - 2)) + 1 > 0$.

When $n < k^2 - 2$ and $n > 0$, we have that $n(n - (k^2 - 2))$ is a negative integer, (n and k are positive integers, so $(n - (k^2 - 2))$ is a negative integer), meaning that -1 is an upper bound for its value, so $n(n - (k^2 - 2)) + 1 \le 0$; the inequality is not satisfied. However, when $n \ge k^2 - 2$, $n(n - (k^2 - 2))$ is a nonnegative integer, for $(n-(k^2-2))$ is now a nonnegative integer, meaning that $n(n-(k^2-2))+1>0$. The inequality is satisfied in this case, so $k^2-2 \leq n \leq k^2$.

Hence, the integer solutions n to the inequality $n^2 \leq k^2 n < n^2+2n+1$ are $n = k^2-2, k^2-1, k^2$, and $n=0$ (don't forget that 0 obviously works!). Since $x=\sqrt{n}$, the possible values for x are $0, \sqrt{k^2-2}, \sqrt{k^2-1}$ and k, meaning that for all positive integers $k \geq 2$, $f(k)=4$. Our answer is $2+24*4=98 \Rightarrow$ D.

§ 25 Problem (naman12)

What is the sum of the digits of the positive integer n such that the number

$$\frac{1! \cdot 2! \cdot 3! \cdots 2019! \cdot 2020!}{n!}$$

is a perfect square?

(A) 2 **(B)** 4 **(C)** 8 **(D)** 10 **(E)** 12

§ 25.1 Solution

Note that $1! \cdot 2! = (1!)^2 \cdot 2$, and so on. Say $a \equiv b$ if the square-free portions of a, b are the same. Then $1! \cdot 2! \cdot 3! \cdots 2019! \cdot 2020! \equiv 2 \cdot 4 \cdot 6 \cdots 2020 \equiv 2^{1010} \cdot 1010! \equiv 1010!$ so $n = 1010$ works. Thus the answer is 2, corresponding to answer choice A.

We now prove that no other n work. Note that we require $1010! \equiv n!$ and since 1009 is a prime, we must have $n \geq 1009$. By Bertrand's postulate, if p is the largest prime that divides $n!$ then $\nu_p(n!) = 1$. Since 1013 is a prime, we must have $1009 \leq n < 1013$. Testing all possible n shows that $n = 1010$ is the only solution.