2021

JMC 12

REMAINS OPEN UNTIL THE DUE DATE

****Administration On An Earlier Date Is Not Even Possible****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL. PLEASE READ THE MANUAL BEFORE February 1, 2021.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 12 CERTIFICATION FORM (found in the Teachers' Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be mailed by trackable mail to the AMC office no later than 24 hours following the exam.
4. *The publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

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*The **January Math Competitions** are brought to you by:*

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and the Mathematical Advancement Comimttee

MAC JMC

January Math Competitions

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JMC 12

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INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL THE TIMER STARTS.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer clearly, edits in submission will not be accepted.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.
9. When you finish the exam, sign your name in the space provided on the Answer Form.

The Mathematical Advancement Comimttee (MAC) reserves the right to re-examine students before deciding whether to grant official status to their scores. Students, regardless of their JMC 12 score, will be invited to the July Invitational Mathematics Examination (JIME).

1. What is the value of $2020^2 - 2021^2 - 2120^2 + 2121^2$?

   (A) 0 **(B)** 100 **(C)** 101 **(D)** 200 **(E)** 201

2. A day in the format *mm/dd* is called *binary* if all of the digits are either 0s or 1s with leading zeros allowed. How many days in a year are binary?

   (A) 5 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 16

3. A mixture has 96 grams of aluminum and 4 grams of barium. Nir the chemist uses magic to remove some aluminum. Now, exactly 90% of the mixture consists of aluminum. How many grams of the mixture now remain?

   (A) 25 **(B)** 40 **(C)** 50 **(D)** 70 **(E)** 72

4. What is the sum of the real solutions to $(|x+1| - |x-1|)^2 = x + 3$?

   (A) $-\frac{3}{4}$ **(B)** 0 **(C)** $\frac{1}{4}$ **(D)** 1 **(E)** $\frac{5}{4}$

5. A positive integer is *pretentious* if it has both even and odd digits. For example, 74 and 83 are pretentious. How many pretentious three-digit numbers are odd?

   (A) 325 **(B)** 375 **(C)** 400 **(D)** 450 **(E)** 775

6. Let $ABCD$ be a square with sides of length 4. Point X is on side CD and point Y is on side BC such that $AX = 5$ and angle AYX is right. What is $AY \cdot XY$?

   (A) 10 **(B)** $9\sqrt{2}$ **(C)** 12 **(D)** $6\sqrt{5}$ **(E)** $7\sqrt{5}$

7. There exists positive integers k such that $k = 3\gcd(20, k)$. What is the sum of all possible values of k?

   (A) 84 **(B)** 108 **(C)** 120 **(D)** 126 **(E)** 132

8. An angle chosen from $\{1^\circ, 2^\circ, \cdots, 90^\circ\}$ and an angle chosen from $\{1^\circ, 2^\circ, \cdots, 89^\circ\}$ determine two angles of a triangle. What is the probability this triangle is obtuse?

   (A) $\frac{11}{45}$ **(B)** $\frac{22}{45}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{11}{15}$

23. Let B and C be on circle ω with center A and radius 4. Let D be the midpoint of $\overline{AB}$ and let E lie on ray $\overline{AC}$ such that $AE = 6$. If there exists point F on ω such that $FD = FC$ and $FB = FE$, what is the ratio of the perimeter of $\triangle ABC$ to its area?

(A) $\frac{4\sqrt{3}}{3}$ **(B)** $\frac{2\sqrt{15}}{3}$ **(C)** $\frac{5\sqrt{3}}{3}$ **(D)** $\frac{\sqrt{35}}{2}$ **(E)** $\frac{12\sqrt{7}}{7}$

24. How many ordered pairs of positive integers (a, b) with $a \leq 100$ and $b \leq 10$ exist such that neither the numerator nor denominator of the below fraction, when completely simplified (i.e. numerator and denominator are relatively prime), are divisible by five?

$$\frac{4^b + 121^b}{2^a - 3^b}$$

(A) 10 **(B)** 12 **(C)** 375 **(D)** 380 **(E)** 382

25. For all complex numbers z such that $|z| < 1$, suppose that

$$\sum_{j=1}^{\infty} z^{j^2} = \sum_{j \in S} \frac{z^j}{1 - z^j} - \sum_{j \in S'} \frac{z^j}{1 - z^j}$$

where S is a unique subset of $\mathbb{Z}_+$ and S' contains all the elements in $\mathbb{Z}_+$ not in S. What is the sum of the six smallest elements in S?

(A) 36 **(B)** 38 **(C)** 41 **(D)** 42 **(E)** 44

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9. Two lines A and B, have 3 and 4 people respectively. At $t = 1$ minutes and continuing every other minute, the first two people from A join the end of line B at the second-to-last and last positions respectively. At $t = 2$ minute and continuing every other minute, the first two people from line B join the end of line A in a similar fashion. If Emma starts in the front of line A at $t = 0$, at what time, in minutes, will Emma first be first in line A again?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 13 **(E)** 14

10. Suppose that $\sin\alpha$ and $\cos\alpha$ are the two roots of $x^2 - nx + \frac{1}{4}$ for some positive n. What is the value of n?

(A) $\frac{\sqrt{2}}{2}$ **(B)** $\sqrt{2}$ **(C)** $\frac{\sqrt{6}}{2}$ **(D)** $\frac{5}{4}$ **(E)** $\frac{1+\sqrt{3}}{2}$

11. Let $a_0, a_1, a_2, \ldots, a_{2021}$ be a sequence such that $a_0 = 1$ and $a_n = 10^n \cdot a_{n-1} + 1$ for positive integers $n \geq 1$. How many terms of this sequence are divisible by 99?

(A) 0 **(B)** 20 **(C)** 21 **(D)** 91 **(E)** 112

12. If a and b are randomly chosen real numbers between -5 and 5, what is the probability that $\lfloor |a| \rfloor - |\lfloor b \rfloor| = 0$? (Recall that $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

(A) $\frac{1}{20}$ **(B)** $\frac{2}{25}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{9}{50}$ **(E)** $\frac{1}{5}$

13. How many ordered triples of integers (a, b, c) satisfy $2 \leq |a| + |b| + |c| \leq 5$?

(A) 80 **(B)** 122 **(C)** 140 **(D)** 208 **(E)** 224

14. Let a_1, a_2, a_3, a_4 be real numbers strictly greater than 1 such that $\log(3\log_{a_1} a_2)$, $\log(3\log_{a_2} a_3)$, $\log(3\log_{a_3} a_4)$, and $\log(3\log_{a_4} a_1)$ are side lengths of a quadrilateral. If p and q are positive numbers such that all $\log_{a_i} a_j$ (where $1 \leq i, j \leq 4$) are always less than p and always greater than q, what is the minimum possible value of $\frac{p}{q}$?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 27 **(E)** 81

15. Let $\triangle ABC$ be a triangle with circumcenter P and $\angle BAC = 60^\circ$. Suppose line BP intersects AC at point X, and line CP intersects AB at point Y. If $CY = 6$ and $\angle PXY = 15^\circ$, what is the length of XY?

(A) 3 **(B)** 4 **(C)** $3\sqrt{2}$ **(D)** $3\sqrt{3}$ **(E)** 6

16. Two distinct divisors of $6^4 = 1296$ are *mutual* if their difference divides their product. For instance, $(4, 2)$ is mutual as $(4-2) \mid 4 \cdot 2$. Suppose a mutual pair (d_1, d_2) exists where $d_1 = kd_2$ for a positive integer k. What is the sum of all possible k?

(A) 14 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 23

17. If x, y, and z are positive real numbers that satisfy the equation

$$xy + yz + zx = 96(y+z) - x^2 = 24(z+x) - y^2 = 54(x+y) - z^2,$$

what is the value of xyz?

(A) 1720 **(B)** 2720 **(C)** 7560 **(D)** 9600 **(E)** 15120

18. A particle is in a 5×5 grid. Each second, it goes to an adjacent cell and when traveling from a cell to another cell, it takes one of the path(s) with shortest time. The particle starts at cell A and travels to cell B in 3 seconds, to cell C in 4 seconds, and finally back to cell A in 5 seconds. How many possible triples $\{A, B, C\}$ exist?

(A) 56 **(B)** 96 **(C)** 104 **(D)** 136 **(E)** 168

19. Two identical circles ω_a and ω_b with radius 1 have centers that are $\frac{4}{3}$ units apart. Two externally tangent circles ω_1 and ω_2 of radius r_1 and r_2 respectively are each internally tangent to both ω_a and ω_b. If $r_1 + r_2 = \frac{1}{2}$, what is $r_1 r_2$?

(A) $\dfrac{9}{512}$ **(B)** $\dfrac{5}{144}$ **(C)** $\dfrac{1}{18}$ **(D)** $\dfrac{1}{16}$ **(E)** $\dfrac{25}{324}$

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20. Let $f(z) = \frac{(z+\overline{z})^2}{\overline{z}} - (z + \overline{z})$ for all non-zero complex numbers z. If $f(z) = 1 + i\sqrt{3}$, what is the sum of all possible values of z?

(A) $\dfrac{4i\sqrt{3}}{3}$ **(B)** $1 + i\sqrt{3}$ **(C)** 0 **(D)** $(\sqrt{3}-1) + i(1+\sqrt{3})$ **(E)** $\dfrac{2}{3}$

21. An invisible ant and an anteater, at the same constant speed of 1 edge length per second, start at (not necessarily distinct) randomly chosen vertices of a cube. Each second, the ant first pings its location to the anteater, then randomly chooses one of the 3 edges emerging from its vertex to traverse immediately. The anteater traverses the edge on the closest path to the ping at the same time the ant travels. If multiple optimal paths exist, one is randomly chosen. The anteater eats the ant if at some time they are both at the same point, not necessarily a vertex. What is the ant's expected lifespan in seconds?

(A) $\dfrac{41}{16}$ **(B)** $\dfrac{11}{4}$ **(C)** $\dfrac{49}{16}$ **(D)** $\dfrac{27}{8}$ **(E)** $\dfrac{55}{16}$

22. For positive reals $a_1, a_2, \ldots, a_n$, the function $\mathcal{D}(a_1, a_2, \ldots, a_n)$ denotes the difference between the arithmetic mean and the geometric mean of those n numbers. Suppose positive real numbers x, y, and z satisfy

$$\begin{cases} xyz = 1 \\ \mathcal{D}(\sqrt{x}, \sqrt{y}, \sqrt{z}) = \frac{74}{13}, \\ \mathcal{D}(\sqrt[3]{x}, \sqrt[3]{y}) + \mathcal{D}(\sqrt[3]{y}, \sqrt[3]{z}) + \mathcal{D}(\sqrt[3]{z}, \sqrt[3]{x}) = \frac{37}{9}. \end{cases}$$

What is the value of $\mathcal{D}(\sqrt[6]{x}, \sqrt[6]{y}, \sqrt[6]{z})$?

(A) $\dfrac{4}{11}$ **(B)** $\dfrac{3}{8}$ **(C)** $\dfrac{5}{13}$ **(D)** $\dfrac{7}{16}$ **(E)** $\dfrac{4}{9}$