2021

JMC 10

REMAINS OPEN UNTIL THE DUE DATE

Administration On An Earlier Date Is Not Even Possible

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL. PLEASE READ THE MANUAL BEFORE FEBRUARY 2, 2021.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 10 CERTIFICATION FORM (found in the Teachers' Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be mailed by trackable mail to the AMC office no later than 24 hours following the exam.
4. *The publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

The ***January Math Competitions*** *are brought to you by:*

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and the Mathematical Advancement Committee

MAC JMC

January Mathematics Competitions

2nd ANNUAL

JMC 10

January Mathematics Competition 10

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL THE TIMER STARTS.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer clearly, edits in submission will not be accepted.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have 75 minutes to complete the test.
9. When you finish the exam, sign your name in the space provided on the Answer Form.

The Committee on the January Mathematics Competitions (CJMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. Students, regardless of their JMC 10 score, will be invited to the July Invitational Mathematics Examination(JIME).

1. What is the value of
$$\frac{20}{2\cdot 1}-\frac{2+0}{2/1}?$$

(A) 3 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 13

2. There exist irrational numbers $e \approx 2.72$ and $\pi \approx 3.14$. How can $|\pi - |e - |e - \pi|||$ be expressed in terms of π and e?

(A) $\pi - e$ **(B)** $2\pi - 2e$ **(C)** $2e$ **(D)** 2π **(E)** $2\pi + e$

3. A group of 8 people are either honest or liars, where honest people always tell the truth and liars always lie. People $P_1, P_2, \cdots, P_8$ stand in a line, and person P_i calls P_{i+1} a liar where $P_1 = P_9$. Out of these eight people, how many liars are there?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 7

4. A day in the format mm/dd is called *binary* if all of the digits are either 0s or 1s with leading zeros allowed. How many days in a year are binary?

(A) 5 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 16

5. A mixture has 96 grams of aluminum and 4 grams of barium. Nir the chemist uses magic to remove some aluminum. Now, exactly 90% of the mixture consists of aluminum. How many grams of the mixture now remain?

(A) 25 **(B)** 40 **(C)** 50 **(D)** 70 **(E)** 72

6. The sum of the ages of a family equals 78. Fifteen years later, the sum of their ages is equal to 153. How many people are in this family?

(A) 4 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 10

7. For some real x, the area of a square equals $3x + 1$ and the product of the lengths of its diagonals equals $5x + 7$. What is the perimeter of this square?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

24. In cyclic convex hexagon $AZBXCY$, diagonals $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ concur at the circumcenter of the hexagon, and quadrilateral $BCYZ$ has area 1. If the sum of the areas of $\triangle ABC$ and the original hexagon is equal to 3, what is the sum of the areas of quadrilaterals $XZAC$ and $YXBA$?

(A) $\frac{3}{2}$ **(B)** 2 **(C)** $\frac{8}{3}$ **(D)** 3 **(E)** 4

25. How many ordered pairs of positive integers (a, b) with $a \leq 100$ and $b \leq 10$ exist such that neither the numerator nor denominator of the below fraction, when completely simplified (i.e. numerator and denominator are relatively prime), are divisible by five?

$$\frac{4^b + 121^b}{2^a - 3^b}$$

(A) 10 **(B)** 12 **(C)** 375 **(D)** 380 **(E)** 382

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8. A positive integer is *pretentious* if it has both even and odd digits. For example, 74 and 83 are pretentious. How many pretentious three-digit numbers are odd?

(A) 325 **(B)** 375 **(C)** 400 **(D)** 450 **(E)** 775

9. In Malachar, the number system is identical to ours, but all real numbers are written with digits in reverse order. A citizen in Malachar writes $15 \cdot 73$. What does this Malacharian write as the answer?

(A) 1095 **(B)** 1887 **(C)** 3723 **(D)** 5901 **(E)** 7881

10. Let $ABCD$ be a square with sides of length 4. Point X is on side CD and point Y is on side BC such that $AX = 5$ and angle AYX is right. What is $AY \cdot XY$?

(A) 10 **(B)** $9\sqrt{2}$ **(C)** 12 **(D)** $6\sqrt{5}$ **(E)** $7\sqrt{5}$

11. There exist positive integers k that satisfy $k = 3\gcd(20, k)$. What is the sum of all possible values of k?

(A) 84 **(B)** 108 **(C)** 120 **(D)** 126 **(E)** 132

12. Mihir draws line $y = 2x$ and Nathan draws line $x + y = n$ for an integer n. The two lines divide the region $y \geq x^2$ into four regions, with regions possibly having infinite area. What is the sum of all possible values of n?

(A) 12 **(B)** 15 **(C)** 23 **(D)** 29 **(E)** 32

13. An angle chosen from $1°, 2°, \ldots, 90°$ and an angle chosen from $1°, 2°, \ldots, 89°$ determine two angles of a triangle. What is the probability this triangle is obtuse?

(A) $\frac{11}{45}$ **(B)** $\frac{22}{45}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{11}{15}$

14. For a certain b, the base b numbers

$$24_b, n, 57_b, 72_b, \ldots$$

form an increasing arithmetic sequence in that specific order. Then, what is the value of n, expressed in base 10?

(A) 47 **(B)** 51 **(C)** 63 **(D)** 64 **(E)** 75

15. Let $a_0, a_1, a_2, \ldots, a_{2021}$ be a sequence such that $a_0 = 1$ and $a_n = 10^n \cdot a_{n-1} + 1$ for positive integers $n \geq 1$. How many terms of this sequence are divisible by 99?

(A) 0 **(B)** 20 **(C)** 21 **(D)** 91 **(E)** 112

16. If a and b are randomly chosen real numbers between -5 and 5, what is the probability that $\lfloor |a| \rfloor - |\lfloor b \rfloor| = 0$? (Recall that $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

(A) $\frac{1}{20}$ **(B)** $\frac{2}{25}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{9}{50}$ **(E)** $\frac{1}{5}$

17. One lit lightbulb is 2 units above the top of spherical ball with a radius of 8. The spherical ball, lying atop a flat floor, casts a shadow. What is the area of this shadow?

(A) 100π **(B)** 144π **(C)** 288π **(D)** 576π **(E)** 1024π

18. If x, y, and z are positive real numbers that satisfy the equation

$$xy + yz + zx = 96(y+z) - x^2 = 24(z+x) - y^2 = 54(x+y) - z^2,$$

what is the value of xyz?

(A) 1720 **(B)** 2720 **(C)** 7560 **(D)** 9600 **(E)** 15120

19. Two distinct divisors of $6^4 = 1296$ are *mutual* if their difference divides their product. For instance, $(4, 2)$ is mutual as $(4-2) \mid 4 \cdot 2$. Suppose a mutual pair (d_1, d_2) exists where $d_1 = kd_2$ for a positive integer k. What is the sum of all possible k?

(A) 14 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 23

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20. A particle is in a 5×5 grid. Each second, it moves to an adjacent cell and when traveling from a cell to another cell, it takes one of the paths with shortest time. The particle starts at cell A and travels to cell B in 3 seconds, to cell C in 4 seconds, and finally back to cell A in 5 seconds. How many possible triples $\{A, B, C\}$ exist?

(A) 56 **(B)** 96 **(C)** 104 **(D)** 136 **(E)** 168

21. Two identical circles ω_a and ω_b with radius 1 have centers that are $\frac{4}{3}$ units apart. Two externally tangent circles ω_1 and ω_2 of radius r_1 and r_2 respectively are each internally tangent to both ω_a and ω_b. If $r_1 + r_2 = \frac{1}{2}$, what is $r_1 r_2$?

(A) $\frac{9}{512}$ **(B)** $\frac{5}{144}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{16}$ **(E)** $\frac{25}{324}$

22. Let r_1, r_2, r_3, r_4 be the roots of $P(x) = x^4 + 4x^3 - 3x^2 + 2x - 1$. Suppose $Q(x)$ is the monic polynomial with all six roots in the form $r_i + r_j$ for integers $1 \leq i < j \leq 4$. What is the coefficient of the x^4 term in the polynomial $Q(x)$?

(A) 32 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 56

23. An invisible ant and an anteater, at the same constant speed of 1 edge length per second, start at (not necessarily distinct) randomly chosen vertices of a cube. Each second, the ant first pings its location to the anteater, then randomly chooses one of the 3 edges emerging from its vertex to traverse immediately. The anteater traverses the edge on the closest path to the ping at the same time the ant travels. If multiple optimal paths exist, one is randomly chosen. The anteater eats the ant if at some time they are both at the same point, not necessarily a vertex. What is the ant's expected lifespan in seconds?

(A) $\frac{41}{16}$ **(B)** $\frac{11}{4}$ **(C)** $\frac{49}{16}$ **(D)** $\frac{27}{8}$ **(E)** $\frac{55}{16}$